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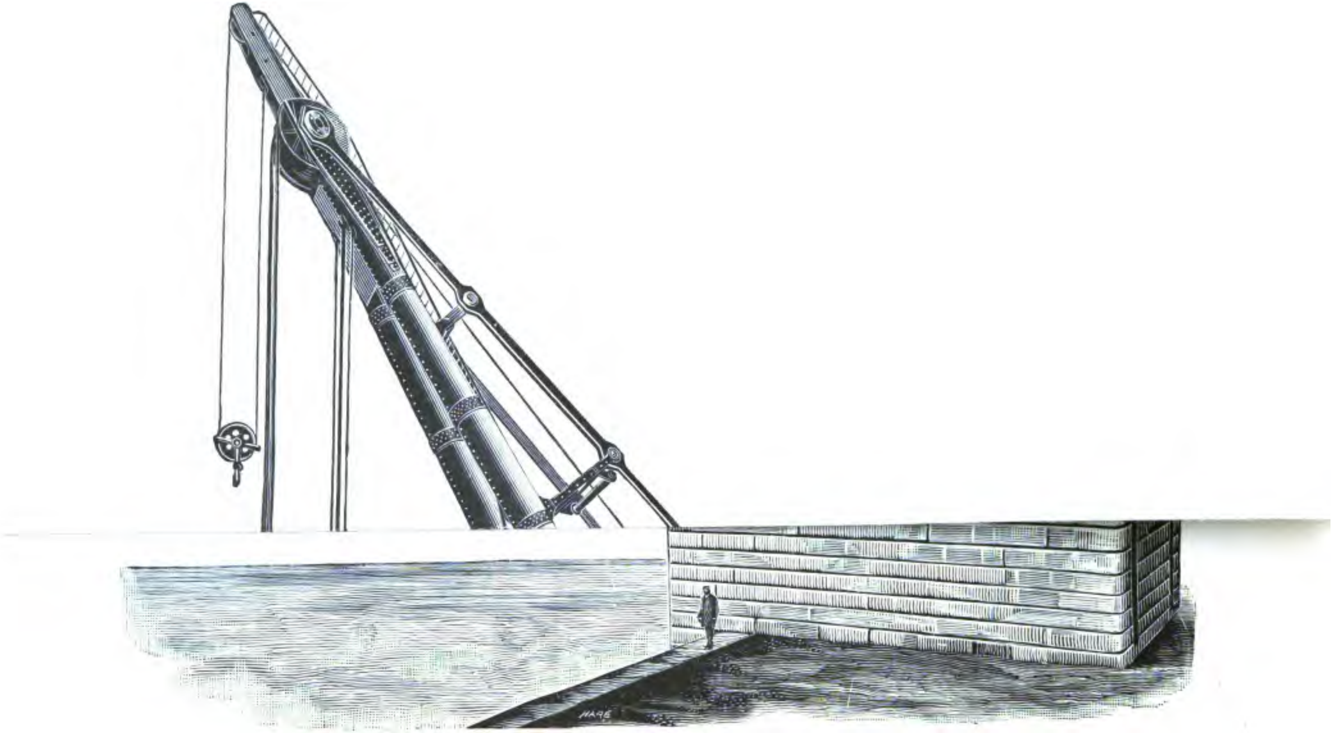
A TEXT-BOOK

ON

APPLIED MECHANICS.

VOLUME II.



130-TON STEAM CRANE, ERECTED AT GLASGOW HARBOUR.

(*For Description, see Lecture* XXVII.)

***Frontispiece.*]**

A TEXT-BOOK

ON

APPLIED MECHANICS.

SPECIALLY ARRANGED
FOR THE USE OF SCIENCE AND ART,
CITY AND GUILDS OF LONDON INSTITUTE,
AND OTHER ENGINEERING STUDENTS.

BY

ANDREW JAMIESON, M.INST.C.E.,
PROFESSOR OF ELECTRICAL ENGINEERING IN THE GLASGOW AND WEST OF SCOTLAND
TECHNICAL COLLEGE; MEMBER OF THE INSTITUTE OF ELECTRICAL ENGINEERS;
FELLOW OF THE ROYAL SOCIETY, EDINBURGH; AUTHOR OF TEXT-BOOKS
ON STEAM AND STEAM-ENGINES, ELEMENTARY APPLIED
MECHANICS, MAGNETISM AND ELECTRICITY,
ELECTRICAL RULES AND TABLES, ETC.

VOLUME II.

With numerous Diagrams and Examination Questions.

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PREFACE TO VOLUME II.

THIS Text-Book has been written expressly for Second and Third Year Students of Applied Mechanics. It, therefore, forms a suitable companion to the Author's *Text-Book on Steam and Steam Engines.* It also forms a direct continuation of his *Elementary Manual on Applied Mechanics;* for it covers the Advanced Stage of the Science and Art Departments Examinations, and treats on many points demanded by the Honours Section. It will, moreover, be found of considerable use to those who aim at passing the Advanced and Honours Stages of the same Examinations in Machine Construction and Drawing, as well as the Examinations of the City and Guilds of London Institute in Mechanical Engineering. At the same time, the treatment of the subject is sufficiently general to satisfy the wants of other engineering students, who do not happen to have these Special Examinations in view.

The book has been divided into six parts:—

I. The Principle of Work and its Applications.
II. Gearing.
III. Motion and Energy.
IV. Graphic Statics.
V. Strength of Materials.
VI. Hydraulics and Hydraulic Machinery.

Parts I. and II. were issued as Volume I., and the remaining Parts now form Volume II. This volume consists of Lectures XX. to XXXIV. under the following general headings:—Velocity and Acceleration—Motion and Energy—Energy of Rotation and Centrifugal Force—Engine Governors and other Applications of Centrifugal Force—Framed Structures—Roof Frames—Deficient Frames—Cranes—Beams and Girders—Stress and Strain, and Bodies under Tension—Strength of Shafts—Strength of Beams and Girders—Deflection of Beams and Girders—Hydraulic Appliances—Refrigerating Machines.

In each Part special reference has been made to the latest and best books, and to papers read before leading Engineering Societies.

In each Part a number of examples have been fully worked out, and at the end of each Lecture a series of carefully-selected questions has been arranged, in the precise order of, and relating solely to, the subject matter of the Lecture, so that Teachers and Students may have a minimum of trouble in finding suitable examples.

Volume I. having been so kindly received, and having already passed into a Second Edition, it has been considered advisable to issue this Volume in time for the coming session, many Teachers having expressed a wish to have the continuation of the work at once. Later, the Author hopes to have the opportunity of still further amplifying and extending Part VI.

In conclusion, he has to thank many of his old Students and friends in connection with the production of the work; more especially, for the help which he received from Mr. Robert M. Anderson with Parts I. to III.; Mr. John H. A. MacIntyre with Part IV.; and Mr. John Anderson with Part V. He has also to thank Mr. Alexander H. Weddell and Mr. John S. Nicholson for preparing numerous drawings, and the various firms who supplied illustrations of their mechanical appliances. Finally, he has received much assistance from Mr. David A. Ramsay, Mr. J. Fred. Nielson, and Mr. David Robertson, Jun., in preparing the manuscript and revising the proofs.

Great care has been taken to avoid errors, but if any should be observed by readers, the Author will be glad to have them pointed out, and to receive any suggestions tending to increase the usefulness of this book.

ANDREW JAMIESON.

The Glasgow and West of Scotland
Technical College,
September, 1897.

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VOLUME II.

PART III.—MOTION AND ENERGY.

LECTURE XX.

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DEFINITION.—**A body is said to be in Motion when it is continually changing its position in space, and to be at Rest when it retains a fixed position in space.**

These are the definitions of *absolute* motion and *absolute* rest. We can never know the absolute motion of any body because we know no fixed bodies to which we may refer its positions at different times. We, therefore, can only deal with the *relative motion* of a body.

DEFINITION.—**A body is said to have Relative Motion with respect to another body when it is continually changing its position relatively to that body.**

Thus, take the case of a train moving on a railway. We always consider its motion relatively to some part of the earth's surface. But the train is carried round the earth's axis and also round the sun by the rotation of the earth itself. And this is not all, for we have reason to believe that the sun itself is not fixed in space but is in motion. A passenger in the train might be at rest relative to the train but he would be in motion relatively to the houses, trees, &c., which the train passed on its way.

Motions of Translation and Rotation.—The motion of a body may be either *Translatory* or *Rotary*.

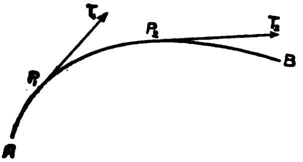
A body is said to have a motion of *simple translation* when all points in the body move with the same velocity and in the same direction at the same instant, so that no line in the body changes its direction. Hence, the motion of the whole body is known when that of any point in it is known.

A body is said to have a motion of *simple rotation* when the various points in the body describe circles about some fixed axis either within or without the body. Hence, the motion of the whole is known when that of any *line* in the body (other than the axis about which the motion takes place) is known.

The motion of a body may be *complex;* being composed or compounded of motions of translation and rotation. Thus, the connecting-rod of an engine has a complex motion. It has a motion of translation in a vertical plane containing the centre line of the engine, and a motion of rotation in the same plane about the crosshead pin.

Definition.—The Path of a moving point is the line, straight or curved, which passes through all the successive positions of the point.

Direction of Motion.—The direction of motion of a body is, at any particular instant, the tangent to the path of the body at that instant, or the path itself if the motion is rectilinear.



ILLUSTRATING DIRECTION OF MOTION.

Thus, let AB be the path of a moving body. When the body occupies the position, P_1, its direction of motion is along P_1, T_1, the tangent to the path at that point. Similarly, when the body occupies the position P_2, its direction of motion is along the tangent P_2, T_2.

Hence, when a body moves in a circular path its direction of motion at any instant will be perpendicular to the radius drawn to its position on the circle at that instant.

Definition.—The Velocity of a body is the rate at which it changes its position.

A *velocity* is completely specified when we know (1) its *direction*, and (2) its *magnitude*.

Hence, a velocity can be completely represented by a straight line of finite length with a suitably-directed arrow head.

Definition.—A body is said to be moving with Uniform Velocity when it is moving in a constant direction and passes over equal distances in equal intervals of time, however small these may be.

The last clause in the above definition is necessary, because a body *might* describe equal distances in equal times, and yet its motion might not be uniform. Thus, a train may describe 20 miles in each of two consecutive hours, and yet its motion may have varied continuously during that time; sometimes its velocity may be 60 miles an hour, and at other times it may be nil.

Uniform Velocity, how Measured.—When uniform, the velocity of a body is measured by its displacement in unit time. Thus:—

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}}.$$

DEFINITION.—A body is said to have Unit Velocity when it describes unit distance in unit time.

The unit of distance in this country is the *foot*, and the unit of time is usually the *second*, although engineers often take the *minute*, or even the *hour*, as the unit of time. For example, the speed of a railway train is always spoken of as so many *miles per hour*, and that of the piston of an engine as so many *feet per minute*.

Whatever units may be used, we get :—

$$\left.\begin{array}{ll} & v = \dfrac{s}{t} \\ \text{Or,} & s = v\,t \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(I)}$$

Where, s = Displacement, or distance described, in time, t.

And, v = Velocity, supposed to be uniform.

*[From the above definition and equation it is evident that v must be the same however small t may be. Thus, let the displacement be very small, say $\Delta\, s$, then the time taken to describe it will be correspondingly small, say $\Delta\, t$, and we get :—

$$v = \frac{\Delta\, s}{\Delta\, t}.$$

This being true for the smallest fraction of time, it must also be true in the limit.

$$\left.\begin{array}{ll} \therefore & v = \dfrac{d\,s}{d\,t} \\ \text{Or,} & d\,s = v\,d\,t \end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad \text{(II)}\ ]$$

DEFINITION.—A body is said to be moving with Variable Velocity when it is either changing its direction of motion or passing over unequal distances in equal intervals of time.

---

* **Students who have no knowledge of the notation of the** ***Calculus*****, and** those merely reading for examination in the Advanced Stage of this subject, may omit for the present the text within the brackets, thus [ ].

From this definition it appears that a body has a *variable velocity* when the direction or magnitude of its velocity is variable. Thus, a point on the rim of the flywheel of an engine has a variable velocity whether the rotary motion of the wheel be uniform or not.

This follows at once from the fact that a velocity is only completely specified when we know its *direction* and *magnitude*, and a change in either the direction or in the magnitude causes a change in the velocity. It is usual, however, in most problems, to speak of the velocity as being uniform or variable, according as the *magnitude* of the velocity is uniform or variable.

Variable Velocity, how Measured.—When variable, the velocity of a body is measured at any particular instant by the displacement which the body would have received if it moved for a unit of time with the same velocity which it had at the instant under consideration.

Thus, we see a train approaching a station and say that its velocity is 10 miles an hour, although we at the same time observe that its velocity is diminishing rapidly, and will soon be zero. By the expression "10 miles an hour" we, therefore, do not mean that it will run 10 miles during the next hour, but simply that *if* the train continued to run for one hour with the same speed that it had at the instant the remark was made, it would travel a distance of 10 miles.

Average Velocity.—When the velocity of a body is variable, and we know its magnitudes for several positions of the body, then its *average* velocity can be found in the same way as we find the average of a series of numbers.

Thus, let v_1, v_2, v_3, ... v_n denote the velocities at n different points in its path; then:—

$$\textbf{Average velocity} = \bar{v} = \frac{v_1 + v_2 + v_3 + \ldots + v_n}{n}.$$

Or it may be defined as follows:—

DEFINITION.—**When a body moves through a certain distance with a variable velocity, its average velocity is that uniform velocity which it would require to have in order to traverse the same distance in the same time.**

$$\left.\begin{aligned} \text{Therefore,}\quad \bar{v} &= \frac{s}{t} \\ \text{Or,}\quad s &= \bar{v}\,t \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (\mathrm{I}_a)$$

If the velocity increase or decrease uniformly, then the mean or average velocity is half the sum of the initial and final velocities.

$$\text{Or,}\quad \bar{v} = \frac{v_1 + v_2}{2}. \quad \ldots\ldots\ldots \quad (\mathrm{III})$$

Where v_1 and v_2 denote the initial and final velocities respectively.

DEFINITION.—**The acceleration of a body is its rate of change of velocity.**

Acceleration may be either uniform or variable.

DEFINITION.—**Acceleration is uniform when equal changes of velocity take place in equal intervals of time, however small these may be.**

Otherwise, the acceleration is variable.

Acceleration, how Measured.—Uniform acceleration is measured by the change in the velocity in a unit of time.

Variable acceleration is measured at any particular instant by what would be the change of velocity in a unit of time, on the supposition that during that unit of time the acceleration remained the same as at the instant under consideration.

If the student thoroughly understands the method of measuring a variable velocity, he should have no difficulty in perceiving from the above statement how variable acceleration is measured.

Uniformly Accelerated Motion.—We shall now deduce the ordinary formulæ for the motion of a body uniformly accelerated in its line of motion.

Let v_1 = Velocity of body at end of time t_1,
„ v_2 = „ „ „ t_2,
„ s = Distance described during interval $(t_2 - t_1)$,
„ a = Acceleration per unit time.

Then, *Change of velocity* $= v_2 - v_1$.

$\therefore$ *Rate of change of velocity* $= \dfrac{v_2 - v_1}{t_2 - t_1}$.

But, *Rate of change of velocity* = *acceleration.*

$$\therefore \qquad a = \frac{v_2 - v_1}{t_2 - t_1}.$$

Or, denoting the interval of time $(t_2 - t_1)$ by t, we get:—

$$\left.\begin{aligned} a &= \frac{v_2 - v_1}{t} \\ \text{Or,} \qquad v_2 &= v_1 + at \end{aligned}\right\} \quad \ldots\ldots \quad \text{(IV)}$$

That is:—**Final Velocity = Initial Velocity + Change of Velocity.**

Again, since the acceleration is uniform, we get:—

$$\textit{Average velocity} = \frac{v_1 + v_2}{2}$$

$$\therefore \qquad s = \frac{v_1 + v_2}{2} \times t.$$

But, $$v_2 = v_1 + a\,t,$$

$$\therefore \qquad s = \frac{v_1 + (v_1 + a\,t)}{2} \times t.$$

Or, $$s = v_1 t + \tfrac{1}{2} a\,t^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(V)}$$

In many problems the time, t, is not given, and we require to find one of the four quantities, s, a, v_1, v_2, having given the other three. From equations (IV) and (V) the following relation between these quantities can easily be deduced by eliminating t. Thus:—

From equation (IV), $a = \dfrac{v_2 - v_1}{t}$,

From equation (V), $s = \dfrac{v_1 + v_2}{2} \times t$.

Multiplying together the corresponding sides of these equations and equating the products, we get:—

$$a\,s = \frac{v_2^2 - v_1^2}{2},$$

$$\left.\begin{array}{ll} \therefore & v_2^2 - v_1^2 = 2\,a\,s \\ \text{Or,} & v_2^2 = v_1^2 + 2\,a\,s \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(VI)}$$

The above formulæ are true for all cases of *uniformly increasing* or *uniformly decreasing* velocity; but in the latter case, the acceleration will be negative, and a must be preceded by the *minus* sign.*

If the body start from rest, that is, if the time, t, be reckoned from the commencement of the motion, then, the initial velocity, $v_1 = 0$, and we get, from the above equations:—

$$v = a\,t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{IV}_a)$$

$$s = \tfrac{1}{2} a\,t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{V}_a)$$

$$v^2 = 2\,a\,s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{VI}_a)$$

Where v = velocity at end of time, t.†

---

* There is no need for deducing, or even stating, the corresponding formulæ when the acceleration is negative. The fewer formulæ to be committed to memory the better, and the student should learn to distinguish between positive and negative (increasing or decreasing) acceleration as indicated by difference in sign, and to supply the proper sign where necessary.

† The general formulæ (IV), (V), and (VI) should be used in all cases. When the body starts from rest, substitute $v_1 = 0$.

Graphical Methods.—Equations (III) and (IV) can be very easily represented by means of a diagram. We may here remark that diagrams of velocities, accelerations, &c., are very useful in assisting the student to answer many problems on the motion of a body, and in what follows we shall have several instances of their use when dealing with the moving parts of engines. Before explaining the following diagrams, it is necessary to remind the student that a velocity, or an acceleration, can be completely represented by a straight line. We have already seen that a *velocity* may be represented by a finite straight line. But an *acceleration* is a change of velocity per unit time. Hence, an acceleration may also be represented by a finite straight line. In the meantime, we are not concerned with the *direction* of the velocity or acceleration, so that the lines representing these may be drawn in any convenient direction.

Velocity and acceleration diagrams are constructed in a way similar to those representing work, viz., by drawing two axes at right angles, along which the velocities or accelerations and intervals of time may be plotted.

Diagram for Uniform Velocity.—Let v = velocity, supposed to be uniform, and t = time. Draw the line A B, along which intervals of time have to be plotted. Thus, let A B represent t. From A, set up A C at right angles to A B, and let A C represent the velocity, v. Complete the rectangle A B D C. Then, clearly, the area of A B D C represents the displacement during the time, t. Thus:—

$$\text{Displacement} = s = v\,t$$
$$= \text{area A B D C.}$$

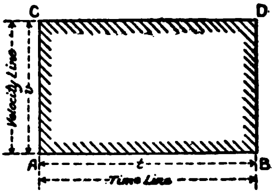


DIAGRAM FOR UNIFORM VELOCITY.

The area, ABDC, represents the displacement in time, t.

$\therefore \quad s = v t.$

Diagram for Uniformly Increasing Velocity.—Let a = the acceleration, and v = velocity at the end of time, t; the initial velocity being zero. As before, let A B represent the interval of time, t. At B, the end of interval t, draw B C to represent v, and join A C. Then, as before, the area of triangle A B C represents the displacement during time, t; since,

$$\textit{Displacement} = s = \textit{mean velocity} \times \textit{time} = \tfrac{1}{2} v \times t$$
$$\text{,,} \qquad = \tfrac{1}{2}\,\text{B C} \times \text{A B} = \text{area A B C.}$$

The velocity at any other time can be found by drawing the ordinate from the point on A B representing the given instant.

Thus, suppose A B represents 4 seconds. Then, the velocity at the end of 3 seconds from the beginning of the motion is represented by the ordinate 3 F. Similarly, at the end of the first second, the velocity is represented by the ordinate 1 D. But in this case, the velocity at the end of the first second is a measure of the acceleration; therefore, 1 D represents the acceleration.

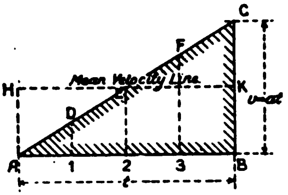


Diagram for Velocity Increasing Uniformly from 0 to v.

The area, A B C, represents the displacement in time, t.

∵ $s = \frac{1}{2} v t.$

If the acceleration be given instead of the final velocity, v, then the diagram can be set out in the following manner :—

Let A 1 represent a unit of time. Draw 1 D at right angles to A B to represent the acceleration, a. Join A D and produce it. Then A C is the velocity line. From this it will be seen that $v = \text{B C} = a\,t.$

$$\begin{aligned} \therefore \quad s &= \tfrac{1}{2} a t^2 \\ &= \tfrac{1}{2} a t \times t \\ &= \tfrac{1}{2} \text{B C} \times \text{A B} \\ &= \text{area A B C.} \end{aligned}$$

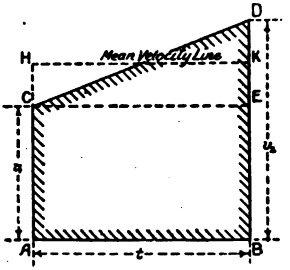


Diagram for Velocity Increasing Uniformly from v_1 to v_2.

The area, A B D C, represents the displacement in time, t.

∵ $s = v_1 t + \frac{1}{2} a t^2.$

If the body does not start from rest let the initial velocity be v_1, and the final velocity, v_2. Then, at each end of A B, the line representing t, draw the ordinates A C and B D to represent v_1 and v_2 respectively, and join C D.

$$\text{Here, } \textit{Displacement} = s = \tfrac{1}{2}(v_1 + v_2) \times t = \tfrac{1}{2}(\text{A C} + \text{B D}) \times \text{A B}$$

$$\text{,,} \qquad = \text{area A B D C.}$$

Also, $\quad \text{E D} = \textit{Change of velocity in time, } t = a\,t.$

And, $\quad \text{B D} = \text{B E} + \text{E D} = v_1 + a\,t.$

$$\therefore \quad s = \tfrac{1}{2}(v_1 + \overline{v_1 + a\,t}) \times t = v_1 t + \tfrac{1}{2} a t^2.$$

We have not drawn the corresponding diagrams for the case when the acceleration is *negative*, but the student should have no difficulty in doing this for himself. Thus, when a is negative, the last diagram would be drawn with the velocity line sloping in the opposite direction.

Motion Due to Gravity.—The most familiar instance of uniformly accelerated motion is that of a body falling under the influence of gravity. Experiments show that if a body be allowed to fall freely in *vacuo* its motion will be uniformly accelerated, and this acceleration is the same for every body (large or small, heavy or light) at the same locality. The letter g is always used to denote this acceleration. Its value depends on the distance of the falling body from the centre of mass of the earth, and varies inversely as the square of this distance. Hence, g is different at different latitudes, being greatest at the poles and least at the equator. When the units of distance and time are the foot and the second, the value of g at the poles is about 32·255, and 32·091 at the equator. Its value at the sea level in the latitude of London is about 32·19, and is generally taken at 32·2 for any place in the British isles.

Formulæ for the motion of bodies under the action of gravity alone are derived from those previously given for uniformly accelerated motion by substituting g for a. Thus:—

(I) *When let fall without initial velocity.*

$$v = g\,t \quad \ldots\ldots\ldots \quad (\mathrm{IV}_b)$$

$$s = \tfrac{1}{2} g\,t^2 \quad \ldots\ldots\ldots \quad (\mathrm{V}_b)$$

$$v^2 = 2 g\,s \quad \ldots\ldots\ldots \quad (\mathrm{VI}_b)$$

(II) *When let fall with initial velocity*, v_1.

$$v_2 = v_1 + g\,t \quad \ldots\ldots\ldots \quad (\mathrm{IV}_c)$$

$$s = v_1 t + \tfrac{1}{2} g\,t^2 \quad \ldots\ldots \quad (\mathrm{V}_c)$$

$$v_2^2 = v_1^2 + 2 g\,s \quad \ldots\ldots\ldots \quad (\mathrm{VI}_c)$$

If the body is thrown upwards with an initial velocity, v_1, then the acceleration due to gravity will be in the opposite direction to that of the motion; consequently, we must make *either* v_1 or g *negative*, according as we consider the downward or the upward direction to be *positive*. In such a case it is usual to make g negative. The rule usually observed is to take the acceleration *positive* or *negative* according as the motion is *increased* or *decreased*.

[**General Formulæ for Linear Motion.**—We have already seen that the velocity of a body is expressed generally as:—

$$v = \frac{d\,s}{d\,t}.$$

We can now give similar expressions for the acceleration when this varies according to any law whatever. Thus, at any instant of time let the velocity of the body be v, and at the end of an interval of time, Δt, let it be $v + \Delta v$, then :—

$$\textit{Acceleration} = \frac{\text{Change of velocity}}{\text{Time required}},$$

$$\quad" \quad = \frac{(v + \Delta v) - v}{\Delta t} = \frac{\Delta v}{\Delta t}.$$

This being true, however small Δt may be, it is, therefore, true in the limit, hence :—

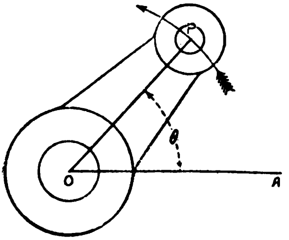
$$a = \frac{dv}{dt}.$$

But,

$$v = \frac{ds}{dt},$$

$$\therefore \quad a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}. \quad \ldots\ldots \quad \text{(VII)}]$$

Body Rotating about an Axis.—Angular Velocity.—We have already said that the motion of a body rotating about an axis is completely known when that of any line in the body, other than the axis of rotation, is known. It is most convenient to take this line passing through the axis, and perpendicular to it. Thus, let O be the intersection of the axis with the plane of the paper, O P a line in the body perpendicular to the axis through O. Then the motion of the body is known when that of the line O P is known. The motion of the line O P is measured by the angle which it describes round the point, O, in unit time. This angle is then spoken of as the **angular velocity** of the body. Hence the following :—



To illustrate Angular Velocity.

Definition.—The angular velocity of a body about an axis is the rate of the angular displacement of any line in the body perpendicular to that axis.

Angular velocity, like linear velocity, may be either *uniform* or *variable*, according as equal or unequal angles are described in equal intervals of time.

Uniform Angular Velocity.—Let the centre line, O P, of the crank in the above figure sweep out the angle, A O P = θ, in the interval of time, t; then the angular velocity of the body (usually denoted by the Greek letter ω) is :—

$$\omega = \frac{\theta}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII)}$$

The angle, θ, is measured in *circular units*, and not in degrees. The unit angle in circular measure is called the *radian*, and may be defined as *the angle subtended at the centre of a circle by an arc of its circumference, equal in length to the radius of the circle.* Hence, if t is in seconds, the *unit of angular velocity* will be the *radian per second.*

Since the length of the arc subtending a right angle is $\frac{\pi}{2} \times r$, and, therefore, the circular measure of a right angle equal to $\frac{\pi}{2}$ radians, we may easily determine the number of *degrees* in a radian. Thus :—

Degrees in 1 *radian* : *Degrees in* 1 *right angle* = 1 : $\frac{\pi}{2}$.

$$\therefore \qquad \text{Degrees in 1 radian} = \frac{90}{\frac{\pi}{2}} = \frac{180}{3{\cdot}1416} = 57{\cdot}29.$$

In general, if θ be the circular measure of an angle of $n°$, then :—

$$n° : 90° = \theta : \frac{\pi}{2},$$

Or,
$$\theta = \frac{n\,\pi}{180}.$$

Hence, if the angle described in time, t, by O P, be $n°$, we get:—

$$\omega = \frac{n\,\pi}{180\,t}. \quad . \quad . \quad . \quad . \quad \text{(VIII}_a\text{)}$$

When the *linear* velocity of any point, P, in the body, and its distance from the axis are known, the angular velocity of the body can be found. Thus :—

Let v = Component of linear velocity of P perpendicular to O P (see the previous figure).

,, r = Radius, O P.

Then, v = Arc described by P in unit time.

$\therefore$ $\frac{v}{r}$ = Circular measure of angle described by O P in *unit time.*

$$\left.\begin{aligned} &\text{i.e.,} && \omega = \frac{v}{r} \\ &\text{Or,} && v = \omega r \end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(IX)*}$$

Variable Angular Velocity.—Angular Acceleration.—When the angular velocity is variable, it is measured in a way similar to that of variable linear velocity.

[Let $\Delta \theta$ = small angle described by O P, in small interval of time, Δt; then we have :—

$$\omega = \frac{\Delta \theta}{\Delta t},$$

which, in the limit, becomes :—

$$\omega = \frac{d\theta}{dt}. \quad \ldots\ldots\ldots \quad \text{(X)]}$$

DEFINITION.—**The angular acceleration of a rotating body is the rate of change of its angular velocity.**

Angular acceleration may be either *uniform* or *variable* according as equal changes of angular velocity take place in equal or unequal intervals of time. When uniform, angular acceleration is measured by the increase or decrease of angular velocity per unit time.

Let ω_1, ω_2 = Angular velocities at the beginning and end of interval of time, t.

„ θ_1, θ_2 = Angular displacements at the beginning and end of interval of time, t.

„ α = Angular acceleration.

$$\left.\begin{aligned} &\text{Then,} && \alpha = \frac{\omega_2 - \omega_1}{t} \\ &\text{Or,} && \omega_2 = \omega_1 + \alpha t \end{aligned}\right\} \quad \ldots\ldots \quad \text{(XI)}$$

From these equations and those previously deduced for uniformly accelerated linear motion, the student will notice the similarity of the relations between the terms s, v, and a, and θ, ω, and α respectively.

Hence, we get the remaining and corresponding equations for rotary motion, viz. :—

$$\theta = \omega_1 t + \tfrac{1}{2} \alpha t^2 \quad \ldots\ldots \quad \text{(XII)}$$

$$\omega_2^2 = \omega_1^2 + 2 \alpha \theta. \quad \ldots\ldots\ldots \quad \text{(XIII)}$$

---

* It is sometimes convenient to speak about the angular velocity of a *point*, such as P in the foregoing figure. Such a phrase is not strictly correct, and when used, it should be understood to mean the angle described in unit time by the radius drawn through the point, P.

[Generally, we have :—

$$\alpha = \frac{d\,\omega}{d\,t} = \frac{d^2\,\theta}{d\,t^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(XIV)]}$$

Composition and Resolution of Velocities.—A moving body may have at any instant two or more velocities in different directions, and it then becomes an important problem to be able to determine the resultant velocity, both in magnitude and in direction. Thus, the magnitude and direction of the motion of a man who walks across the deck of a moving ship is different from that of the ship and also from that of his motion relative to the deck. Similarly, the motion of a point on the rim of a carriage wheel in motion is, in general, different in magnitude and direction from its circular motion about the axle, and also from the onward motion of the wheel as a whole.

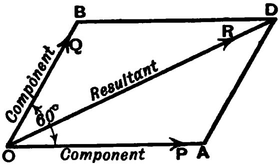
The process of finding a single velocity equivalent in effect to two or more velocities is called the *Composition of Velocities.*

The process of finding two or more velocities equivalent in effect to a single velocity is called the *Resolution of Velocities.*

DEFINITIONS.—**The single velocity which is equivalent to two or more velocities is called their Resultant, and these two or more velocities are called the Components.**

Parallelogram of Velocities.—If two component velocities be represented, in magnitude and direction, by two adjacent sides, O A, O B, of a parallelogram, their resultant velocity will be represented by the diagonal, O D, through their intersection.

Thus, if a moving point, O, possess simultaneously two velocities, P and Q, in directions O A and O B respectively, and, if O A and O B represent the magnitudes of these velocities, their resultant velocity, R, will be represented both in magnitude and in direction by the diagonal, O D, of the parallelogram constructed on O A, and O B, as adjacent sides.



PARALLELOGRAM LAW.

Let θ = angle between the directions of the velocities, P and Q.

„ $\alpha = \angle$ A O D, and $\beta = \angle$ B O D, the angles between the direction of the resultant, R, and the components P and Q respectively.

Then the student may easily prove from Euclid II., 13 and 14, or by trigonometry, that:—

$$R^2 = P^2 + Q^2 + 2\,P\,Q\cos\theta. \qquad \text{(XV)}$$

$$\left.\begin{aligned} &\text{And,} \quad \tan\alpha = \frac{Q\sin\theta}{P + Q\cos\theta} \\ &\text{Or,} \quad \tan\beta = \frac{P\sin\theta}{Q + P\cos\theta} \end{aligned}\right\} \qquad \text{(XVI)}$$

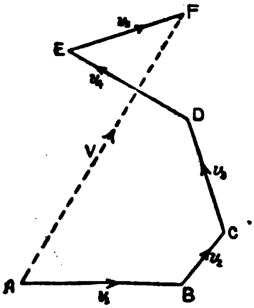
From these equations the magnitude and direction of the resultant velocity can be calculated.

It is not necessary to complete the parallelogram as explained above, it being quite sufficient to draw but one-half of the figure. Thus, A D is equal and parallel to O B; hence, as much can be determined from the triangle, O A D, as from the complete parallelogram, O A D B.

Triangle of Velocities.—If two component velocities be represented in magnitude and direction by two sides of a triangle taken in order, their resultant will be represented in magnitude and direction by the third side taken in the reverse direction.

Hence, if there be simultaneously impressed on a point three velocities represented in magnitude and direction by the sides of a triangle taken in order, then the point will remain at rest.*

Polygon of Velocities.—If several component velocities be represented by all but one of the sides of a polygon, A B C D E F, taken in order—the resultant velocity will be represented in magnitude and direction by the remaining side, A F, taken in the opposite direction.



POLYGON OF VELOCITIES.

Thus, if a moving point have simultaneously impressed upon it velocities, v_1, v_2, . . v_5, and these are represented in magnitude and direction by the sides A B, B C, . . . E F of a polygon, A B C D E F, then the resultant velocity will be represented in magnitude and direction by the side, A F, required to complete the polygon.

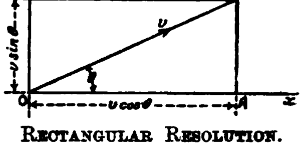
---

* In setting out the *Parallelogram*, or *Triangle of Velocities*, it is not necessary to draw the sides parallel to the velocities represented. The sides may be drawn in directions *perpendicular* to the respective velocities, or, indeed, at any other angle, so long as the angle is the same for all the sides. In such cases the line representing the resultant will be equally inclined to its true direction.

If the figure whose sides represent the component velocities be closed or completed when the last velocity has been represented, then there is no resultant velocity, and the point will remain at rest.

It is equally important to be able to *resolve* a given velocity into two or more component velocities. Thus, the velocity, R (see the figure for *Parallelogram of Velocities*), can be resolved into two components, P and Q, in the directions O A, O B respectively. Or, the velocity, V (in the last figure), may be resolved into a number of components, v_1, v_2, . . ., in directions A B, B C, . . . Further, the directions of the component velocities may be anything we like. Thus, in resolving a given velocity, R, into two components, we can do so in an infinite number of ways, since an infinite number of parallelograms, such as O A D B, can be found having O D for one of their diagonals. When, however, the *directions* of the components are fixed, their magnitudes will be definite and easily determined. Referring to the figure for the *Parallelogram of Velocities*, let O D represent a velocity, R, which has to be resolved into two components in the directions O A and O B. From D draw D A parallel to B O and D B parallel to A O, meeting the lines O A and O B in the points A and B respectively. Then O A and O B represent the component velocities P and Q to the same scale that O D represents the velocity R.

The most important case of resolution is that wherein the given velocity has to be resolved into components whose directions are at right angles to each other. Thus, let it be required to resolve the velocity, v, whose direction is O C, into its **Rectangular Components** along O x and O y.



RECTANGULAR RESOLUTION.

From C drop the perpendiculars C A, C B on the axes O x and O y. Then, O A, O B are the components in the required directions.

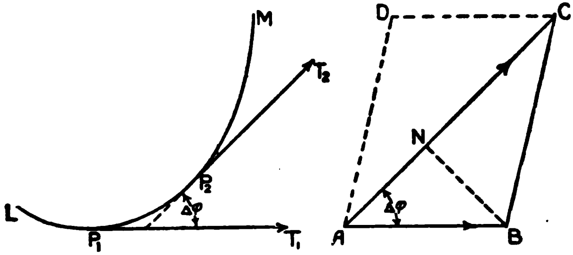
Let v_x, v_y = Components of v in directions O x, O y respectively.
 „ θ = Angle between the directions of v and v_x.

$$\left.\begin{array}{ll}\text{Then,} & v_x = v\cos\theta \\ \text{And,} & v_y = v\sin\theta\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XVII)}$$

Composition and Resolution of Accelerations.—Since an *acceleration* is a *rate of change of velocity*, whether in *magnitude* or in *direction*, it follows that accelerations may be compounded or resolved according to the same rules as velocities.

If the direction of motion of a body be constant, then change of velocity can only take place in that direction. Thus, if a body is constrained to move in a rectilinear path its only acceleration is one of magnitude, and takes place along the straight line in which the body moves.

Again, the velocity of a body may be constant in *magnitude*, but variable in *direction*, as in the case of a body moving with uniform speed in a circle. Or, it may vary both in magnitude and in direction, as in the case of the bob of a pendulum swinging to and fro. The **Total Acceleration**, in any case, may be found in the following manner:—



Total Acceleration of a Moving Body.

Let L M be the path of a moving body, and P_1, P_2 its positions at the beginning and end of an interval of time, *t*.

At P_1, its velocity is in the direction of the tangent, $P_1 T_1$, and at P_2, its velocity is in the direction, $P_2 T_2$.

From A draw A B and A C to represent in magnitude and direction the velocities of the body at the points P_1 and P_2 respectively. Join B C, and complete the parallelogram, A B C D. Then A C represents the resultant velocity whose components are A B and A D or B C. But, if the velocity of the body had remained constant in magnitude and in direction during the time, *t*, its velocity at the end of that interval of time would have been represented by A B. Hence, in the above case, A D, or B C, represents, in magnitude and direction, the *change of velocity* during the time, *t*.

$$\therefore \qquad \textbf{Total acceleration} = \frac{BC}{t}.$$

[Suppose the arc $P_1 P_2$ to be very small; and

Let v = Velocity of body at point, P_1.
„ $v + \Delta v$ = Velocity of body at point, P_2.
„ Δt = Small interval of time required to traverse the small arc, $P_1 P_2$.
„ $\Delta \varphi$ = Angle between tangents to curve at P_1 and P_2.

From B draw B N perpendicular to A C. Then B N and N C represent respectively the components of the total acceleration, B C, along lines normal and tangential to the curve at a point near to P_1.

Hence, *Normal Acceleration* = limit of $\frac{BN}{\Delta t}$

And, *Tangential Acceleration* = limit of $\frac{NC}{\Delta t}$.

Now, $BN = AB \sin \Delta \varphi = v \sin \Delta \varphi.$

$\therefore$ Limit of $\frac{BN}{\Delta t} = v \frac{d\varphi}{dt}$.

In the limit, let ds denote the infinitesimally small arc, $P_1 P_2$, and let ρ denote the radius of curvature at the point, P_1 or P_2.

Then, $$v\frac{d\varphi}{dt} = v\frac{ds}{dt} \cdot \frac{d\varphi}{ds} = v^2\frac{d\varphi}{ds}.$$

But, from the properties of plane curves, we know that :—

$$\frac{d\varphi}{ds} = \frac{1}{\rho}.^*$$

$$\therefore \qquad \text{Normal Acceleration} = \frac{v^2}{\rho} \quad . \quad . \quad . \quad . \quad \text{(XVIII)}$$

Again, $$\text{Limit of } \frac{NC}{\Delta t} = \text{limit of } \frac{\Delta v}{\Delta t} = \frac{dv}{dt}.$$

$$\therefore \qquad \text{Tangential Acceleration} = \frac{dv}{dt} \quad . \quad . \quad . \quad . \quad \text{(XIX)}$$

The result expressed in equation (XIX) agrees with the corresponding general equation previously deduced.

In the case of a body moving in a circle with uniform motion, we get $\rho = r$ = radius of circle, and v is constant. Then the tangential acceleration is *nil*, and the

$$\text{Normal or Radial Acceleration} = \frac{v^2}{r}. \quad (\text{XVIII}_a)$$

This is usually spoken of as the *Centripetal Acceleration.*]

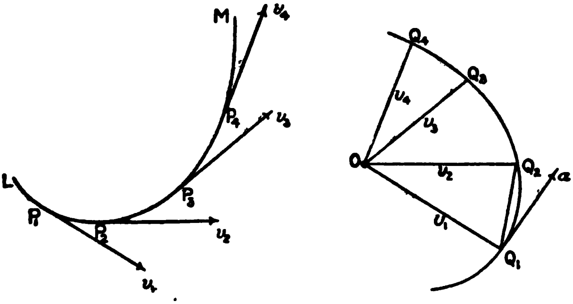
The Hodograph—Uniform Motion in a Circle.—We shall now extend the foregoing principles to the determination of the acceleration of a body which moves with uniform velocity in a circle. In the first place we shall briefly describe the properties of the *Hodograph.*

---

* See Todhunter's *Diff. Calculus*, p. 343.

Definition.—If a point, P, be moving in any manner in a straight or curved path, and if from a fixed point, straight lines be drawn representing in magnitude and direction the velocities of P at different points of its path, the locus of the extremities of those lines will be a curve which is the Hodograph of P's motion.

Thus, let L M be the path of a moving point, P. Let the velocities at the points P_1, P_2, P_3 . . . be v_1, v_2, v_3 . . . From any point, O, draw O Q_1, O Q_2, O Q_3 . . . respectively parallel to v_1, v_2, v_3 . . . and of lengths representing these velocities. Then the curve, Q_1, Q_2, Q_3, Q_4, which is the *locus* of the point Q, is the hodograph of P's motion in the path, L M. Hence, to every point on the curve, L M, there will be a corresponding point on the hodograph, so that while the body describes the curve, L M, we may imagine a point to describe the hodograph.



The Hodograph.

We shall now prove the following property of the hodograph:—

The acceleration of the body at any point on the curve, L M, is represented in magnitude and direction by the velocity of the corresponding point on the hodograph.

Let v = Average velocity between P_1 and P_2.
„ Δt = Indefinitely small time required to describe arc $P_1 P_2$.

Then, $$v = \frac{P_1 P_2}{\Delta t}.$$

But O Q_1, O Q_2, represent the velocities of the body at the beginning and end of the interval of time, Δt. Therefore, chord $Q_1 Q_2$ represents the *change of velocity* of the body, during that interval of time.

That is, $$\left.\begin{array}{l}\textit{Acceleration of body between}\\ \quad P_1 \textit{ and } P_2\end{array}\right\} = \frac{Q_1 Q_2}{\Delta t}.$$

But, in the limit, when P_2 approaches indefinitely near to P_1, and, therefore, also Q_2 approaches indefinitely near to Q_1, we get:—

$$\text{Chord } Q_1 Q_2 = \text{Arc } Q_1 Q_2.$$

But, $$\frac{\text{Arc } Q_1 Q_2}{\Delta t} = \textit{Velocity of } Q \textit{ in hodograph.}$$

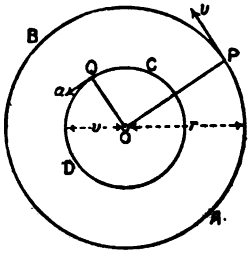
$$\therefore \left.\begin{array}{l}\textbf{Acceleration of}\\ \textbf{body in curve}\\ \textbf{L M}\end{array}\right\} = \textbf{Velocity of Q in hodograph.}$$

Again, since the direction of motion of a point on a curve is along the tangent to the curve at that point, so the direction of motion of Q on the hodograph at any point is along the tangent to the hodograph at that point. Hence, the *direction* of the acceleration of the moving body at any point on the curve, L M, is represented by the tangent at the corresponding point on the hodograph.

Thus, let P_1 and Q_1 be corresponding points on the path and hodograph respectively. Then, O Q_1 represents the velocity of the body at P_1, and the tangent to the hodograph at Q_1 represents the *direction* of the acceleration at the same point.

When a body describes a circle with uniform velocity, it is evident that there can be no tangential acceleration.

Let A P B represent the circular path of a body moving with uniform velocity, v. Then, it is clear that the hodograph of the moving body will also be a circle whose radius is v. With centre, O, and radius representing v, describe a circle, C Q D. Then, circle C Q D is the hodograph. Let P be the position of the body at any instant. Draw the radius, O Q, of the hodograph parallel to the tangent at P; or, what is the same thing, draw O Q perpendicular to O P. Since the radius, O P, describes equal angles in equal times, it follows at once that the radius, O Q, of the hodograph will also describe equal angles in equal times. In other words, the velocity of Q in the hodograph is uniform. Now, the magnitude and direction of the *velocity* of Q represent the magnitude and direction of the *acceleration* of P. Therefore, the direction of the acceleration of P is that of the tangent to the hodograph at the point, Q; that is, it



HODOGRAPH FOR UNIFORM MOTION IN A CIRCLE.

is along the radius, P O. As to the magnitude of this acceleration, we observe that:—

$$\textit{Acceleration of } \mathrm{P} = a = \textit{Velocity of } \mathrm{Q}.$$

Since Q describes the circle, C Q D, in the same time, *t*, as P describes the circle, A P B, we get:—

$$\text{For hodograph,} \quad t = \frac{\textit{Circumference of } \mathrm{C\,Q\,D}}{\textit{Velocity of } \mathrm{Q}} = \frac{2\pi \times v}{a} \quad . \ (1)$$

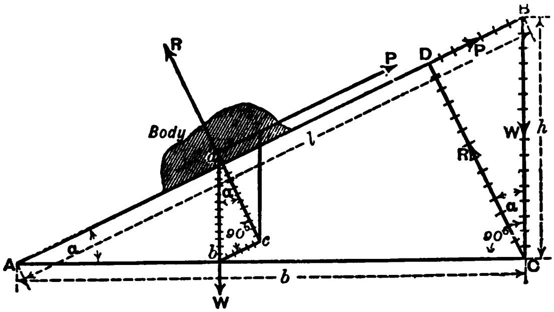
$$\text{For Path of P,} \quad t = \frac{\textit{Circumference of } \mathrm{A\,P\,B}}{\textit{Velocity of } \mathrm{P}} = \frac{2\pi \times r}{v} \quad . \ (2)$$

Equating (1) and (2) we get:—

$$\frac{2\pi v}{a} = \frac{2\pi r}{v}.$$

$$\therefore \quad \textbf{Acceleration of P} = a = \frac{v^2}{r}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad \text{(XX)}$$

Example I.—A body slides down a smooth inclined plane, determine its velocity at the foot of the plane. If the plane has a rise of 25 per cent., what distance would a body, descending along it from a state of rest, describe in five seconds? Find also the time occupied in sliding down the first 50 feet of the length of the plane.



Motion on Smooth Inclined Plane.

Answer.—Let a = Acceleration of the body along the plane.
„ α = Inclination of plane to the horizon.

(1) If the body were free to move vertically downwards its acceleration in that direction would be g. But since it is con-

strained to move in the direction B A, its acceleration in this direction will be less, being, in fact, the component of g, along B A.

Hence, resolve g into its rectangular components in directions along and at right angles to B A. Then :—

Acceleration along B A $= g \sin \alpha$.

Acceleration perpendicular to B A $= g \cos \alpha$.

The latter component has no effect on the motion of the body. Hence :—

$$\textbf{Acceleration down the plane} = a = g \sin \alpha . \quad . \quad . \quad (1)$$

Let t = Time required to slide along a length, s.
" v = Velocity at the end of time, t.

Then, from equation (IV_a) $v = a\,t$.

$\therefore \quad v = g\,t \sin \alpha . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$

From equation (V_a) $s = \frac{1}{2} a\,t^2$.

$\therefore \quad s = \frac{1}{2} g\,t^2 \sin \alpha . \quad . \quad . \quad . \quad . \quad . \quad (3)$

And from equation (VI_a) $v^2 = 2\,a\,s$.

$\therefore \quad v^2 = 2\,g\,s \sin \alpha$.

But, $s \sin \alpha$ = Height of plane of length, s,
" = h, say.

Then, $v^2 = 2\,g\,h . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$

That is,—*The velocity acquired by a body in sliding down a smooth inclined plane is the same as that acquired by a body falling freely through a distance equal to the height of the plane.*

From the given data, we get :—

$$\sin \alpha = \frac{25}{100} = \cdot 25,$$

$$t = 5 \text{ seconds.}$$

$\therefore$ From equation (3), we get :—

$$s = \tfrac{1}{2} g\,t^2 \sin \alpha = \tfrac{1}{2} \times 32{\cdot}2 \times 5 \times 5 \times \cdot 25$$

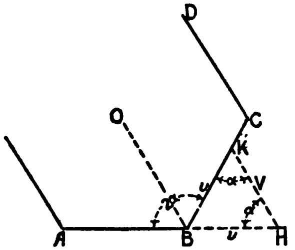
$$\text{,,} = \mathbf{100{\cdot}625 \text{ feet.}}$$

(2) Here, $s = 50$ feet, and we require t.
From equation (3), we get :—

$$s = \tfrac{1}{2} g\,t^2 \sin \alpha.$$

$$\therefore \quad t = \sqrt{\frac{2\,s}{g \sin \alpha}} = \sqrt{\frac{2 \times 50}{32{\cdot}2 \times \cdot 25}} = \mathbf{3{\cdot}52 \text{ seconds.}}$$

Example II.—State the rule for the composition of two velocities. If a particle describes the perimeter of a regular polygon with a constant velocity, v, show that there must be impressed on it, at each angular point a velocity equal to $2\,v\cos\alpha$, directed towards the centre of the circumscribing circle, where α denotes half an angle of the polygon. (S. & A. Hons. Theor. Mechs. Exam., 1885.)



To Illustrate Example IV.

Answer.—For answer to first part, see *Parallelogram of Velocities.* Let A B C D . . represent the sides of a regular polygon, whose centre is O. When the particle arrives at B, its direction of motion is suddenly changed from A B to B C, while the magnitude of the velocity remains unaltered. To find the magnitude and direction of the velocity which must have been imparted to the particle at the point, B, we may proceed as follows:—

Produce A B, and set off B H, to represent the velocity, v, of the particle along A B, and B K along B C, to represent the velocity in that direction. Then H K represents in magnitude and direction the change of velocity which must have been imparted to the particle at the point, B. The magnitude of this velocity can be found from the triangle, B H K, or equation (XV).

$$\text{For,}\quad V^2 = v^2 + v^2 - 2\,v^2\cos(180^\circ - 2\,\alpha) = 2\,v^2(1 + \cos 2\,\alpha)$$

$$\text{,,}\quad = 4\,v^2\cos^2\alpha. \quad [\text{Since } 1 + \cos 2\,\alpha = 2\cos^2\alpha.]$$

$$\therefore\quad \mathbf{V = 2\,v\cos\alpha.}$$

Join B to O, the centre of the polygon, and we get:—
In triangle B H K;

$$\text{Exterior} \angle ABC = \angle BHK + \angle BKH.$$

$$\text{But,}\quad \angle BHK = \angle BKH,$$

$$\therefore\quad \angle BHK \text{ or } \angle BKH = \tfrac{1}{2}\angle ABC = \alpha.$$

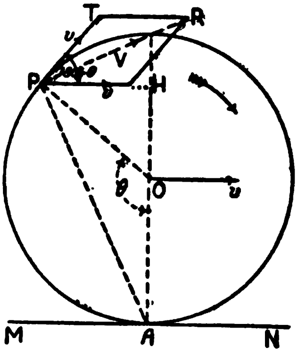
$$\therefore\quad HK \text{ is parallel to } BO, \text{ since } BO \text{ bisects } \angle ABC.$$

Therefore, the velocity impressed on the particle at B is directed along B O, towards the centre.

Example III.—Find, at any instant, the magnitude and direction of the velocity of a point on the rim of a wheel which rolls along a road with a constant speed, v.

Answer.—Take any point, P, on the rim of the wheel, and let the radius, drawn through P, make an angle, θ, with the vertical radius, O A.

Then, since the centre of the wheel is moving with velocity, v, it follows that the tangential velocity of any point on the rim is also v. This is, however, but one of the components of the actual velocity of P. The actual velocity of P is the resultant of two velocities—viz., v, along the tangent at P, and v, horizontally, since the point, P, in addition to moving round O, as a centre, is also being carried in a horizontal direction along with the wheel as a whole. By completing the parallelogram of velocities, as shown, the resultant velocity, V, can be found. The angle between the component velocities is 180° − θ. Hence, from equation (XV) :—



Velocity of a Point on a Rolling Wheel.

$$V^2 = v^2 + v^2 + 2\,v^2 \cos(180^\circ - \theta).$$

$$\therefore \qquad ,, = 2\,v^2\{1 + \cos(180^\circ - \theta)\} = 4\,v^2 \sin^2\frac{\theta}{2}.$$

$$\therefore \qquad V = 2\,v \sin\frac{\theta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Next, as to the direction of the resultant velocity, V.

Since P R bisects the angle between P T and P H,

$$\therefore \qquad \angle RPH = 90^\circ - \frac{\theta}{2}.$$

But, $\angle OAP = \frac{1}{2}\angle POH = 90^\circ - \frac{\theta}{2}.$

$$\therefore \qquad \angle RPH = \angle OAP.$$

In the $\triangle APH$, $\angle OAP$ = complement of $\angle APH$.

$\therefore$ $\angle RPH$ = ,, ,,

$\therefore$ $\angle RPA$ is a right angle.

Hence, the direction of motion of P is perpendicular to the line joining P with A, the point of the wheel which is in contact with the ground.

The point, A, is called the **Instantaneous Centre** of motion for all points on the rim of the wheel; because any point, such as P, is moving at any instant on the circumference of a circle having A for its centre and A P as its radius.

The direction of the actual motion of any point, P, is, at any instant, inclined to the horizontal at an angle equal to $90° - \frac{\theta}{2}$.

EXAMPLE IV.—In the previous example find the magnitude and direction of the actual velocity of the point, P, when the radius, O P, makes angles of 0°, 90°, 180°, and 270° with the vertical radius, O A. Also, find the position of P when the resultant velocity, V, is equal to v.

ANSWER.—(1) *When* $\theta = 0°$. From equation (1), Example III., we get:—

$$V = 2v \sin\frac{\theta}{2} = 0, \text{ since } \sin 0° = 0.$$

i.e., the point is at rest when it is in contact with the ground A.

(2) *When* $\theta = 90°$. Here $\sin\frac{\theta}{2} = \sin 45° = \frac{\sqrt{2}}{2}$.

$$\therefore \qquad V = 2v \sin\frac{\theta}{2} = 2v \times \frac{\sqrt{2}}{2} = v\sqrt{2}.$$

$$\text{Also, } \left.\begin{array}{l}\text{Direction which}\\ \text{V makes with}\\ \text{the horizon}\end{array}\right\} = 90° - \frac{\theta}{2} = 45°.$$

(3) *When* $\theta = 180°$. $\sin\frac{\theta}{2} = \sin 90° = 1$.

$$\therefore \qquad V = 2v \sin\frac{\theta}{2} = 2v \times 1 = 2v.$$

$$\text{And, } \left.\begin{array}{l}\text{Direction which}\\ \text{V makes with}\\ \text{the horizon}\end{array}\right\} = 90° - \frac{\theta}{2} = 0°.$$

That is, when P is vertically over O, it is moving horizontally with a velocity equal to twice the speed of the wheel.

(4) *When* $\theta = 270°$. $\sin\frac{\theta}{2} = \sin 135° = \frac{\sqrt{2}}{2}$.

$$\therefore \qquad V = 2v \sin\frac{\theta}{2} = 2v \times \frac{\sqrt{2}}{2} = v\sqrt{2}.$$

$$\text{And, } \left.\begin{array}{l}\text{The inclination}\\ \text{of V to the}\\ \text{horizon}\end{array}\right\} = 90° - 135° = -45°.$$

(5) *To find θ when* $V = v$.

Here, $$V = 2v \sin\frac{\theta}{2}.$$

$$\therefore \quad \sin\frac{\theta}{2} = \frac{V}{2v} = \tfrac{1}{2}.$$

$$\therefore \quad \frac{\theta}{2} = 30^\circ, \text{ or } 150^\circ,$$

$$\therefore \quad \theta = 60^\circ, \text{ or } 300^\circ.$$

These agree with the two positions of P when the chord, A P, is equal to the radius, O A.

LECTURE XX.—QUESTIONS.

1. Define the terms velocity and acceleration, and explain how these are measured—(1) when uniform; (2) when variable. Give examples of bodies having accelerations—(*a*) constant in magnitude and direction; (*b*) constant in magnitude but not in direction; (*c*) constant in direction but not in magnitude; (*d*) variable both in magnitude and direction.

2. When the velocity of a particle is uniformly accelerated, show that $s = \frac{1}{2} a t^2$. A particle moves from a state of rest under the action of a force which increases its velocity by 20 feet per second in every second of its motion. After four seconds the force ceases to act on it. What distance does it describe in the first six seconds of its motion? (S. & A. Adv. Theor. Mechs. Exam., 1880.) *Ans.* 320 feet.

3. Establish the formulæ for uniform acceleration in the direction of motion:—$v_2 = v_1 + a\,t$; $s = v_1\,t + \frac{1}{2}\,a\,t^2$, and from these results deduce the formula—$v_2^2 = v_1^2 + 2\,a\,s$. Find an expression for the distance described in the *n*th second.

4. A cage is ascending the shaft of a mine at a uniform rate of 10 feet per second. When it is 50 feet from the top the speed is diminished, so that it now moves with a uniformly retarded velocity, and finally comes to rest at the top. Find the retardation. *Ans.* 1 foot per second.

5. State the rule for the composition of two velocities. Draw two lines, A B, A C, containing an acute angle. A particle is at A moving with a given velocity, V, from A towards B. Give a construction for determining the velocity that must be impressed on it, to make it move with a velocity, 2 V, from A towards C. (S. & A. Adv. Theor. Mechs. Exam., 1894.)

6. A particle describes the perimeter of a regular hexagon with a constant velocity of 100 feet a second. Find the magnitude and direction of the velocity that must be communicated to it, at the instant it reaches an angular point. (S. & A. Adv. Theor. Mechs. Exam., 1889.) *Ans.* 100 feet per second towards centre of hexagon.

7. Two bodies start together from rest, and move in directions at right angles to each other. One moves with a uniform velocity of 3 feet per second, while the motion of the other is uniformly accelerated. At the end of four seconds the bodies are found to be 20 feet apart. Determine the acceleration of the latter body. *Ans.* 2 feet per second.

8. Two bodies, P and Q, move with different velocities along the same line. What is the relative velocity of Q to P? If Q is allowed to fall freely, and two seconds after P is allowed to fall freely from the same point, find the relative velocity of Q to P at any subsequent time. (S. & A. Adv. Theor. Mechs. Exam., 1893.) *Ans.* 64·4 feet per second.

9. Define angular velocity. P is a point of a body turning uniformly round a fixed axis, and P N is a line drawn from P at right angles to the axis. If P N describes an angle of 375° in three seconds, what is the angular velocity of the body? and if P N is 6 feet long, what is the linear velocity of P? (S. & A. Adv. Theor. Mechs. Exam., 1892.) *Ans.* (1) $0{\cdot}7\,\pi$ radians per second; (2) $4{\cdot}2\,\pi$ feet per second.

10. A point is describing a circle of radius 21 feet, with a uniform velocity of 12 feet per second. Find the change in its velocity after it has described

one-sixth of a whole circumference. *Ans.* 12 feet per second, at 120° with first direction.

11. A wheel, whose diameter is 5 feet, turns forty times a minute; find its angular velocity and the linear velocity of a point on its circumference. If the centre of the wheel moves in a straight line with a velocity of 20 miles an hour; what are the velocities, relative to a very distant fixed point in the straight line, of the ends of the diameter which is at any instant vertical? (S. & A. Adv. Theor. Mechs. Exam., 1884.) *Ans.* (1) $\omega = \frac{4\pi}{3}$ radians per second; (2) 10·5 feet per second; (3) upper end = 40 miles per hour; lower end = zero.

12. What is the numerical value of the angular velocity of a body which turns uniformly round a fixed axis twenty-five times a minute? A B C is a triangle right angled at C. It is turning with a given angular velocity, ω, round an axis through A, at right angles to its plane. Find the magnitude and direction of the velocities of B and C; and also the relative velocity of B to C. (S. & A. Adv. Theor. Mechs. Exam., 1892.) *Ans.* $\frac{5}{6}\pi$ radians per second.

13. A train descending a gradient increases its speed from 40 to 49 miles per hour in four and a-half minutes. Find the mean acceleration. Taking the acceleration due to gravity at 32 in feet and seconds, determine the gradient. *Ans.* (1) 0·049 foot per second per second, or 120 miles per hour per hour; (2) 1 in 654.

14. Given the base, b, of a smooth inclined plane, find its height, h, so that the horizontal component of the velocity of a body at the foot of the plane shall be a maximum. *Ans.* $h = b$.

15. Define the hodograph, and prove that the acceleration of a point's motion is equal to the velocity with which the hodograph is traced out. Determine, by means of the hodograph, the acceleration of a body which moves with uniform velocity in a circle.

16. Define the angular velocity of a moving point with respect to a fixed point. Under what circumstances will the angular velocity of the moving point be equal to its linear velocity divided by its distance? Draw an equilateral triangle A B C, having each side 12 feet long; a point moves along B C with a velocity of 10 feet a second; when it is at C, what is its angular velocity with respect to A? (S. & A. Adv. Theor. Mechs. Exam., 1896.)

17. Two circles touch each other externally, and the point of contact (A) is in the same vertical line as the centres; from any point (P) of the upper circumference draw a straight line P A Q to meet the lower circumference in Q; if a particle is allowed to fall from P along P Q, show that the time it takes to reach Q is constant for all positions of P. Also compare the times in which P A and A Q are described. (S. & A. Adv. Theor. Mechs. Exam., 1896.)

LECTURE XXI.

CONTENTS.—Quantity of Motion—Definition of Momentum—Example I.—Newton's Laws of Motion—Examples II. and III.—Motion on a Double Inclined Plane—Examples IV. and V.—Energy—Definition of Energy—Definitions of Potential and Kinetic Energy—Expression for Kinetic Energy—Energy Equations—Examples VI., VII., and VIII.—Questions.

Quantity of Motion.—In the preceding Lecture we have confined our attention chiefly to cases of pure motion—that is, motion considered apart from mass and force. In this Lecture we shall treat of the motion of bodies as produced by the action of external forces, and establish the relations between the *quantity of motion* thus produced and the magnitude of the forces producing it. **Quantity of motion** is measured by the product of the *mass* and its *velocity*. The term **Momentum** is used instead of *Quantity of motion*, and hence we get the following:—

DEFINITION.—**The momentum of a moving body is the product of its mass and velocity.**

Thus, let m be the mass, and v the velocity of a body:—

Then, **Momentum** $= m\,v$.

EXAMPLE I.—Of two steam hammers, one weighs 5 tons and the other 10 tons. The former has a drop of 10 feet and the latter 6 feet. Compare their momenta at the end of their respective strokes.

ANSWER.—In order to find their velocities at the moment of impact, we may employ formula (VI_b) of Lecture XX.:—

$$v^2 = 2\,g\,s,$$

$\therefore$ for the first hammer, $v_1 = \sqrt{2 \times 32 \times 10} = 8\sqrt{10}$ ft. per sec.

And, for the second hammer, $v_2 = \sqrt{2 \times 32 \times 6} = 8\sqrt{6}$,,

$$\therefore \left.\frac{\text{Momentum of first hammer}}{\text{Momentum of second hammer}}\right\} = \frac{m_1 v_1}{m_2 v_2} = \frac{5 \times 8\sqrt{10}}{10 \times 8\sqrt{6}} = \frac{\sqrt{5}}{2\sqrt{3}} = \frac{1}{1{\cdot}549}.$$

Newton's Laws of Motion.—The three fundamental laws of Dynamics, called *Laws of Motion*, were first clearly set forth by Newton, and may be stated as follows:—

LAW I. (*Law of Inertia*).—**Every body continues in a state of rest, or of uniform motion in a straight line, except in so far as it may be compelled to change that state by external force acting on it.**

LAW II. (*Law of Force and Motion*).—**Rate of change of momentum is proportional to the force which causes it, and takes place in the direction of the force.**

LAW III. (*Law of Stress*).—**When two bodies mutually act upon each other, the momenta developed in the same time are equal but opposite in direction.**

Or, To every action there is an equal and opposite reaction.

Law I.—This Law asserts that matter is *indifferent* to motion, *i.e.*, has no *innate* tendency to start into motion when at rest, nor to change its motion, either in magnitude or in direction, when once it is made to move.* Hence, a body at rest or in motion, and unacted upon by force, will continue to remain at rest, or to move on in a straight line with uniform motion. Should any change take place in its motion, then we immediately infer that the body has been acted upon by some external force. This tendency of matter to resist change in its state of rest or of uniform motion in a straight line is called **Inertia**, and the first Law is often spoken of as the *Law of Inertia.*

Law II.—The first Law asserts that change of momentum is caused by the action of force, and the second Law gives us a means of measuring this force, viz., that the force is proportional to the *rate of change of momentum.*

Let F = Force producing change of momentum.
,, m = Mass of body.
,, v_1, v_2 = Initial and final velocities of body.
,, t = Time during which F acts.

Then, *Change of momentum* $= m(v_2 - v_1)$.

And, $\left.\textit{Rate of change of momentum}\right\} = \dfrac{m(v_2 - v_1)}{t}.$

∴ By the *Second Law of Motion*, we get:—

$$F \propto \frac{m(v_2 - v_1)}{t}.$$

Or, $$F = \frac{m(v_2 - v_1)}{t} \times \text{constant}.$$

It now remains to establish the exact relation between those quantities. If we accept the definition of *Unit Force* given on page 2, Lecture I., Vol. I., as being *that force which, acting for unit time on unit mass, produces unit change of velocity*, we find the numerical value of the *constant* in the above equation to be *unity*.

i.e., $$F = \frac{m(v_2 - v_1)}{t}.$$

But we have shown in Lecture XX. that:—

$$a = \frac{v_2 - v_1}{t},$$

where a denotes the acceleration produced when the motion is uniformly accelerated.

$$\therefore \quad F = m\,a.$$

The above definition is that of the *Absolute Unit* of Force; and, therefore, the force, F, as given by these equations, is expressed in absolute units. Engineers, however, prefer measuring their forces by the weights which they are capable of supporting, and the above equations may be modified to suit these units. Let P, P_1, be the statical measures of the forces required to produce accelerations, a, a_1, on a given mass, m; then by Law II., we get:—

$$P : P_1 = a : a_1.$$

If one of these forces be that due to gravity, viz., the weight, w, of a body, then the acceleration is g, and we get:—

$$P : w = a : g.$$

Or, $$P = \frac{w\,a}{g} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(I)}$$

This equation expresses the force, P, in the same units as w, and if w be stated in *pounds weight* that will be in what we have previously called *gravitation* units.

Law III.—This Law asserts that when two bodies mutually act upon each other, the momenta generated in each are equal, but in opposite directions. Thus, when a shot is fired from a gun, the force of the explosion produces momentum in the gun equal in amount to that of the shot, and causes the recoil. We shall, however, see later on that the other effects produced in the gun and the shot are not numerically equal. In the case of mutual action between two bodies incapable of relative motion, the Law asserts that they act and react on each other with equal forces. Thus, a weight lies on a table, and presses on it with a certain force; then the table reacts on the weight with an equal and opposite force, so that every action is accompanied by an equal and opposite reaction.

The truth of this Law has been assumed throughout the whole of the preceding parts of this treatise—viz., that the effort exerted between two bodies is always equal to the resistance overcome. The two equal and opposite forces caused by the mutual action between two bodies are together spoken of as a **Stress**, and for this reason the above Law is sometimes called the *Law of Stress.* The subject of *internal stress* will be discussed in another part of this work.

We shall now apply the preceding results to some examples.

EXAMPLE II.—A 40-lb. shot is fired from a 5-ton gun with an initial velocity of 1,500 feet per second. Find the velocity of the gun's recoil, and the mean force of the explosion, supposing the gun to be 10 feet long.

ANSWER.—Let W, w = Weight of gun and shot respectively.

„ V, v = Velocity „ „

(1) By the *Third Law*:—

Momentum of gun = momentum of shot.

$$\therefore \quad W V = w v$$

$$i.e., \quad 5 \times 2240 \times V = 40 \times 1500,$$

$$\therefore \quad V = \frac{40 \times 1500}{5 \times 2240} = \mathbf{5{\cdot}36 \text{ ft. per sec.}}$$

(2) In order to find the mean effort exerted during the explosion of the powder, we must first determine the acceleration of the shot along the muzzle of the gun. Since the gun is 10 feet long, and the velocity of the shot as it leaves the

gun is 1,500 ft. per second, we get, from the formula (VI_a), Lecture XX. :—

$$v^2 = 2as,$$

$$\therefore \quad 1500^2 = 2 \times a \times 10,$$

$$\therefore \quad a = \frac{1500^2}{20} = 112{,}500 \text{ ft. per second per second.}$$

$$\text{But,} \quad P = \frac{w}{g} \times a,$$

$$\therefore \quad P = \frac{40}{32} \times 112{,}500 = \mathbf{140{,}625\ lbs.}$$

Example III.—A railway train, exclusive of engine, weighs 200 tons, and moves on a level line. In 10 minutes its speed is increased from 10 miles per hour to 40 miles per hour. Determine the mean pull between the engine and train, the frictional resistances being taken at 10 lbs. per ton.

Answer.—The pull between the engine and train consists of two parts; (1) the force required to accelerate the train, and (2) the force required to overcome the frictional resistances.

(1) Let P_1 = Force required to accelerate the train,

$$\text{Then,} \quad P_1 = \frac{wa}{g} = \frac{w(v_2 - v_1)}{gt}.$$

$$\text{But,} \quad v_1 = 10 \text{ miles per hour} = \frac{44}{3} \text{ ft. per second.}$$

$$v_2 = 40 \quad \text{,,} \quad = \frac{176}{3} \quad \text{,,}$$

$$\text{And,} \quad t = 10 \text{ minutes} = 600 \text{ seconds.}$$

$$\therefore \quad P_1 = \frac{200 \times 2240 \times \left(\frac{176}{3} - \frac{44}{3}\right)}{32 \times 600} = \mathbf{1026{\cdot}6\ lbs.}$$

(2) The resistance of friction being 10 lbs. per ton,

The *total frictional resistance* = $P_2 = 200 \times 10 = \mathbf{2000\ lbs.}$

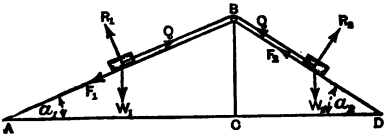
$$\therefore \quad \left.\begin{array}{c}\textbf{Mean pull between}\\ \textbf{engine and train}\end{array}\right\} = P_1 + P_2 = 1026{\cdot}\dot{6} + 2000$$
$$= \mathbf{3026{\cdot}\dot{6}\ lbs.}$$

Motion on a Double Inclined Plane.—Let A B C, D B C, be the two planes placed back to back, and let W_1, be the ascending, and W_2, the descending load, these loads being connected by a weightless rope passing over a frictionless and weightless pulley at B. We require to determine the motion—*i.e.*, the *acceleration* of the bodies, and the tension in the connecting rope.

Let α_1, α_2 = Inclinations of planes ABC, DBC respectively.
,, μ_1, μ_2 = Coefficients of friction between W_1, W_2, and their respective planes.
,, F_1, F_2 = Frictional resistances in the two cases.
,, P_1, P_2 = Effective forces acting on W_1, W_2 respectively in causing motion.
,, Q = Tension in connecting rope.
,, a = Acceleration due to effective forces P_1, P_2.

Then, the effective force causing the *upward* motion of W_1, is :—

$$P_1 = Q - W_1 \sin \alpha_1 - F_1.$$



DOUBLE INCLINED PLANE.

Similarly, the effective force in causing the *downward* motion of W_2, is :—

$$P_2 = W_2 \sin \alpha_2 - Q - F_2.$$

But, $F_1 = \mu_1 W_1 \cos \alpha_1$.

And, $F_2 = \mu_2 W_2 \cos \alpha_2$.

∴ $P_1 = Q - W_1 (\sin \alpha_1 + \mu_1 \cos \alpha_1)$, ... (1)

And, $P_2 = W_2 (\sin \alpha_2 - \mu_2 \cos \alpha_2) - Q$. . . . . (2)

Again, $P_1 = \dfrac{W_1}{g} a$ . . . . (3)

And, $P_2 = \dfrac{W_2}{g} a$. . . . . (4)

To determine the acceleration, a:—

From, (1) + (2),

$$P_1 + P_2 = W_2(\sin\alpha_2 - \mu_2\cos\alpha_2) - W_1(\sin\alpha_1 + \mu_1\cos\alpha_1)$$

From, (3) + (4),

$$P_1 + P_2 = \frac{W_1 + W_2}{g} \times a.$$

$$\frac{W_1 + W_2}{g} \times a = W_2(\sin\alpha_2 - \mu_2\cos\alpha_2) - W_1(\sin\alpha_1 + \mu_1\cos\alpha_1),$$

$$\therefore \quad a = \frac{W_2(\sin\alpha_2 - \mu_2\cos\alpha_2) - W_1(\sin\alpha_1 + \mu_1\cos\alpha_1)}{W_1 + W_2} \times g. \quad \text{(II)}$$

To determine the tension in the rope:—

Equation (1) ÷ (2), $\dfrac{P_1}{P_2} = \dfrac{Q - W_1(\sin\alpha_1 + \mu_1\cos\alpha_1)}{W_2(\sin\alpha_2 - \mu_2\cos\alpha_2) - Q}$,

" (3) ÷ (4), $\dfrac{P_1}{P_2} = \dfrac{W_1}{W_2}$.

$$\therefore \quad \frac{Q - W_1(\sin\alpha_1 + \mu_1\cos\alpha_1)}{W_2(\sin\alpha_2 - \mu_2\cos\alpha_2) - Q} = \frac{W_1}{W_2},$$

$$\therefore (W_1 + W_2)Q = W_1W_2(\sin\alpha_1 + \mu_1\cos\alpha_1 + \sin\alpha_2 - \mu_2\cos\alpha_2)$$

$$\therefore \quad Q = \frac{W_1W_2(\sin\alpha_1 + \sin\alpha_2 + \mu_1\cos\alpha_1 - \mu_2\cos\alpha_2)}{W_1 + W_2} \quad \text{(III)}$$

We shall show how these formulæ are modified to suit some particular cases, but the student should try to prove each particular case independently of the general case just demonstrated.

Case I.—Suppose the planes to be equally inclined to the horizon, and equally rough, so that $\alpha_1 = \alpha_2 = \alpha$, and $\mu_1 = \mu_2 = \mu$; then, from equation (II):—

$$a = \frac{W_2(\sin\alpha - \mu\cos\alpha) - W_1(\sin\alpha + \mu\cos\alpha)}{W_1 + W_2} g,$$

$$\therefore \; a = \frac{(W_2 - W_1)\sin\alpha - \mu(W_2 + W_1)\cos\alpha}{W_1 + W_2} g \quad . \; . \quad (\text{II}_a)$$

From equation (III),

$$Q = \frac{2\,W_1\,W_2\sin\alpha}{W_1 + W_2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (\text{III}_a)$$

CASE II.—Let the planes be equally inclined, and smooth; so that $\alpha_1 = \alpha_2 = \alpha$, and $\mu_1 = \mu_2 = 0$; then, from equation (II):—

$$a = \frac{W_2 \sin \alpha - W_1 \sin \alpha}{W_1 + W_2} g,$$

$$\therefore \quad a = \frac{(W_2 - W_1) \sin \alpha}{W_1 + W_2} g \quad . \; . \; . \; . \; . \; . \quad (II_b)$$

And from equation (III),

$$Q = \frac{2 W_1 W_2 \sin \alpha}{W_1 + W_2} \quad . \; . \; . \; . \; . \; . \; . \quad (III_b)$$

Equations (III_a) and (III_b) show that the degree of roughness of the planes does not affect the tension in the rope, when the planes are equally inclined to the horizon.

CASE III.—Let the plane, AB, be horizontal, and $\mu_1 = \mu_2 = \mu$, and suppose W_2 by falling vertically to drag W_1 along AB. In this case $\alpha_1 = 0$, and $\alpha_2 = 90°$; then, from equation (II):—

$$a = \frac{W_2 (\sin 90° - \mu \cos 90°) - W_1 (\sin 0 + \mu \cos 0)}{W_1 + W_2} g,$$

$$\therefore \; a = \frac{W_2 - \mu W_1}{W_1 + W_2} g \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (II_c)$$

And from equation (III),

$$Q = \frac{W_1 W_2 (1 + \mu)}{W_1 + W_2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (III_c)$$

CASE IV.—In the previous case, let the horizontal plane be smooth, so that, $\mu = 0$:—

Then, $$a = \frac{W_2}{W_1 + W_2} g \quad . \; . \; . \; . \; . \; . \quad (II_d)$$

And, $$Q = \frac{W_1 W_2}{W_1 + W_2} \quad . \; . \; . \; . \; . \; . \; . \quad (III_d)$$

CASE V.—Suppose the weights to be suspended over the frictionless and weightless pulley B, and the parts of the rope to hang vertically.

In this case, $\alpha_1 = \alpha_2 90°$; $\mu_1 = \mu_2 = 0$; then:—

From equation (II),

$$a = \frac{W_2 - W_1}{W_1 + W_2} g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II_e)$$

From equation (III),

$$Q = \frac{2 W_1 W_2}{W_1 + W_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (III_e)$$

These last equations are of great importance to the student of *Theoretical Mechanics*, because they enable him, by means of an Atwood's machine, to determine the value of g, at the place where the experiment is conducted.

Example IV.—A cage weighing 1 ton is being raised from a mine with an acceleration of 10 feet per second. Find the tension in the rope. If a miner, whose weight is 150 lbs., is raised with the cage, find the pressure between him and the cage. Again, if the cage be lowered with the same acceleration, what would then be the tension in the rope, and the pressure between the man and cage?

Answer.—(1) *To find tension in rope during ascent of cage.*

Let W = Weight of cage = 1 ton = 2,240 lbs.
„ w = Weight of man = 150 lbs.
„ Q = Tension of rope in lbs.
„ a = Acceleration of cage = 10 ft. per sec. per sec.

Then, neglecting the weight of the rope, and in the meantime that of the man, we get:—

Effective pull causing motion = P = Q − W.

But, by the *Second Law of Motion*, $P = \frac{W}{g}a$.

$$\therefore \qquad Q - W = \frac{W}{g}a.$$

Hence,

$$Q = W\left(1 + \frac{a}{g}\right)$$

$$\therefore \qquad Q = 2240 \times \left(1 + \frac{10}{32}\right) = \mathbf{2{,}940\,lbs.}$$

That is, the tension in the rope is *greater* than the weight raised by 700 lbs.

If the weight of the miner be taken into account, we must increase W, by 150, and then we get:—

$$Q = 3136{\cdot}9 \text{ lbs.}$$

(2) *To find the pressure between man and cage.*

$$\left.\begin{array}{l}\textit{Pressure between man}\\ \quad\textit{and cage}\end{array}\right\} = \left\{\begin{array}{l}\textit{Weight of man + Force required}\\ \quad\textit{to accelerate his upward motion}\end{array}\right.$$

$$\text{,,} \qquad \text{,,} \qquad = w + \frac{w}{g}a,$$

$$\therefore \qquad w_1 = 150 + \frac{150}{32} \times 10 = 196{\cdot}9 \text{ lbs.}$$

Under these circumstances he will feel heavier by 46·9 lbs.

(3) *To find tension in rope during descent of cage.*

In this case, we get:—

$$\left.\begin{array}{l}\textit{Effective pull causing}\\ \quad\textit{motion}\end{array}\right\} = P = W - Q,$$

$$\text{And,} \qquad P = \frac{W}{g} \times a,$$

$$\therefore \qquad W - Q = \frac{W}{g} \times a,$$

$$\therefore \qquad Q = W\left(1 - \frac{a}{g}\right)$$

$$Q = 2{,}240 \times \left(1 - \frac{10}{32}\right) = 1{,}540 \text{ lbs.}$$

That is, the tension in the rope is *less* than the weight of the cage by 700 lbs.

Similarly, it can be shown that the pressure between the man and the floor of the cage during descent, is 103·1 lbs.; or, 46·9 lbs. less than his real weight.

Example V.—In a double inclined plane, having a rise of 1 in 20, the loaded and empty trucks run on parallel lines of rails, the connection being made by means of two ropes passing round drums at the summit of the plane. Five loaded trucks when descending pull up an equal number of empty ones. Each empty truck weighs 5 cwts., and when loaded carries 20 cwts. of material. The diameter of the drums at the top of the incline is 8 feet, and on the same shaft is fitted a brake pulley 6 feet in diameter. The length of the inclined plane is 1 mile. Taking the coefficient of friction between the trucks and the rails at 20 lbs. per ton, but neglecting other frictional resistances; determine (1) the acceleration of the trucks and their speed at the end of one minute after starting; (2) the tension in the ropes during the free motion of the whole; and (3) the constant frictional resistance which must be exerted at the rim of the brake pulley, during the last three-eighths of the run, in order to just bring the whole to rest at the end of the journey.

Answer.—Using the same letters as in the text.

Let W_1 = Total weight of five empty trucks = 25 cwts.
„ W_2 = „ „ loaded „ = 125 „
„ P_1 = Effective force causing motion of W_1.
„ P_2 = „ „ „ W_2.
„ Q = Tension in ropes.
„ α = Inclination of the plane.
„ μ = Coefficient of friction = 20 lbs. per ton = $\frac{1}{112}$.

Then, $\sin\alpha = \frac{1}{20}$; and since α is small we may assume $\cos\alpha = 1$.

(1) *To find the acceleration of the trucks.*

The effective pull causing the motion of the empty trucks, is:—

$$P_1 = Q - W_1(\sin\alpha + \mu\cos\alpha),$$

$$\therefore \quad P_1 = Q - 25\left(\frac{1}{20} + \frac{1}{112} \times 1\right) = Q - \frac{165}{112}\text{ cwt.} \quad . \quad . \quad (1)$$

The effective pull causing the motion of the loaded trucks, is:—

$$P_2 = W_2(\sin\alpha - \mu\cos\alpha) - Q,$$

$$„ \quad = 125\left(\frac{1}{20} - \frac{1}{112} \times 1\right) - Q = \frac{575}{112} - Q\text{ cwt.} \quad . \quad . \quad (2)$$

Again, by *Second Law of Motion*, we get:—

$$P_1 = \frac{W_1}{g} \times a = \frac{25}{32} \times a \text{ cwt.} \quad . \quad . \quad . \quad . \quad (3)$$

$$P_2 = \frac{W_2}{g} \times a = \frac{125}{32} \times a \text{ cwt.} \quad . \quad . \quad . \quad (4)$$

(1) + (2), $P_1 + P_2 = \frac{575}{112} - \frac{165}{112} = \frac{205}{56}$ cwt.

(3) + (4), $P_1 + P_2 = \left(\frac{25}{32} + \frac{125}{32}\right) \times a = \frac{150}{32} a$ cwt.

$$\therefore \quad \frac{150}{32} \times a = \frac{205}{56}$$

$$\therefore \quad a = 0{\cdot}78 \text{ ft. per sec. per sec.}$$

That is, the trucks move with an acceleration of 0·78 foot per second per second.

At the end of one minute from starting the speed would be:—

$$v = a\,t = {\cdot}78 \times 60 = 46{\cdot}8 \text{ ft. per sec.}$$

Or, at the end of one minute they would be moving with a speed somewhat greater than 30 miles per hour.

(2) *To find tension in the ropes.*

Since we have assumed that the machinery at the top of the incline offers no resistance to the motion, it is evident that the tension in each rope will be the same. Hence:—

(1) ÷ (2), $\frac{P_1}{P_2} = \frac{Q - \frac{165}{112}}{\frac{575}{112} - Q} = \frac{112\,Q - 165}{575 - 112\,Q}.$

(3) ÷ (4), $\frac{P_1}{P_2} = \frac{W_1}{W_2} = \frac{25}{125} = \frac{1}{5}.$

$$\therefore \quad \frac{1}{5} = \frac{112\,Q - 165}{575 - 112\,Q}.$$

$$\therefore \quad Q = \frac{1400}{672} = 2{\cdot}08 \text{ cwt.}$$

(3) *To find the frictional resistance at the rim of the brake pulley in order to bring the trucks to rest at the end of the run.*

Here we have first to obtain the speed of the trucks at the instant when the brake is applied, and then find the retardation or negative acceleration necessary to bring the trucks to rest at the desired place.

The velocity v, of the trucks at the instant when the brake is applied is given by the formula :—

$$v^2 - v_1^2 = 2\,a\,s.$$

Where v_1 = Initial velocity = 0 in this case.
„ a = Acceleration just found = 0·78 ft. per sec. per sec.
„ s = Distance traversed = $\frac{5}{8}$ mile.

The acceleration during the application of the brake may be found by the same formula. In this case, however, the initial velocity is v, and the final velocity is zero.

Let a_1 = Acceleration of the trucks during the application of the brake.
„ s_1 = Distance traversed = $\frac{3}{8}$ mile.

Then, before the brakes are applied :—

$$v^2 - 0^2 = 2\,a\,s$$

Or, $$v^2 = 2\,a\,s.$$

And after the brakes have been applied :—

$$0^2 - v^2 = 2\,a_1\,s_1,$$

Or, $$v_2 = -2\,a_1\,s_1.$$

$$\therefore \quad a_1 = -\frac{a\,s}{s_1} = -\frac{\cdot 78 \times \frac{5}{8}}{\frac{3}{8}} = -\mathbf{1{\cdot}3\ ft.\ per\ sec.\ per\ sec.}$$

(4) *To determine the tensions in the two ropes.*

These will not now be equal as when the motion was free. The tension in the rope coming on to the drum will be much less than before, whilst that on the other rope will be greater.

Let Q_1, Q_2 = Tensions in the ropes attached to the empty and loaded trucks respectively.

Then the effective pull P_1, causing the motion of the ascending trucks is as before :—

$$P_1 = Q_1 - W_1(\sin\alpha + \mu\cos\alpha)$$

But, $$P_1 = \frac{W_1 a_1}{g}$$

$$\therefore \quad Q_1 = W_1\left\{\sin\alpha + \mu\cos\alpha + \frac{a_1}{g}\right\}$$

$$,, \quad = 25\left\{\frac{1}{20} + \frac{1}{112} \times 1 - \frac{1\cdot3}{32}\right\}\text{cwt.},$$

$$,, \quad = \frac{25 \times 20\cdot5}{1120}\text{ cwt.} = 51\cdot25 \text{ lbs.}$$

Similarly, the tension Q_2, in the rope attached to the loaded trucks is,

$$Q_2 = W_2\left\{\sin\alpha - \mu\cos\alpha - \frac{a_1}{g}\right\}$$

$$,, \quad = 125\left\{\frac{1}{20} - \frac{1}{112} \times 1 + \frac{1\cdot3}{32}\right\}\text{cwt.}$$

$$,, \quad = \frac{125 \times 91\cdot5}{1120}\text{ cwt.} = 1143\cdot75 \text{ lbs.}$$

The difference in the tensions in the two ropes is caused by the resistance offered by the brake. Hence, the resultant couple due to this difference in the tension must be balanced by the couple at the brake wheel.

Let F = Frictional resistance at the rim of the brake wheel.
,, R = Radius of the drums = 4 ft.
,, r = Radius of brake wheel = 3 ft.

Then, $$F \times r = (Q_2 - Q_1) \times R,$$

$$\therefore \quad F \times 3 = (1143\cdot75 - 51\cdot25) \times 4,$$

$$\therefore \quad F = 1456\cdot7 \text{ lbs.}$$

Energy.—If we raise a body of W lbs. weight through a vertical height of *h* feet from some given datum level, we confer upon that body the capability of doing work equal to W *h* ft.-lbs. For, in raising the body we expend W *h* ft.-lbs. of work, and if it be allowed to return to its original level it will give out an equal amount of work.

Again, we have seen that if a body be in motion and its speed reduced, some force must have acted upon it in bringing about this change of state. Further, this resisting force must have been overcome through some distance, and, therefore, work is expended. Thus, a body in motion is capable of doing work, the measure of which is the work done against a resisting force or forces in bringing the body to rest.

This capability of doing work which a body possesses in virtue of its position or condition is called **Energy.** Hence, we have the following:—

Definition.—**The energy of a body is its capability of doing work in virtue of its Position, Condition, or Motion.**

It is usual to distinguish between that form of energy due to the position or state of a body, and that due to its motion. To the former the term **Potential** is applied, and to the latter **Kinetic.** This distinction may be stated in the form of a definition.

Definitions.—**Potential Energy is that form of energy which a body possesses in virtue of its Position or Condition.**

Kinetic Energy is that form of energy which a body possesses in virtue of its Motion.

Thus, a raised weight, such as the weight of a clock, or the monkey of a pile-driving engine, has *potential* energy in virtue of its *position.* In the first case the slowly falling weight gives up its stored energy to the mechanism of the clock in overcoming frictional resistances, and thus keeps the clock going, the pendulum being simply a regulator or governor. In the second case, the monkey is allowed to fall freely and its energy is employed in forcing the pile into the ground at the instant of the blow. Similarly, the water in a mill dam possesses potential energy due to its *position* relatively to the water wheel. Again, a stretched helical spring, or coiled spiral spring such as is used in watches and clocks, possesses potential energy due to its *stretched condition.* A lump of coal, or gunpowder, has potential energy in virtue of its *chemical condition;* a magnet has potential energy in virtue of its *magnetic condition;* and the steam in a boiler has potential energy in virtue of its *heat condition,* and so on.

When the monkey of the pile driver is at the top of its stroke its energy is entirely in the potential form. When it is descending it is evident that its potential energy is rapidly decreasing whilst its kinetic energy is increasing.

Neglecting frictional and other resistances, the **Principle of the Conservation of Energy** (see Lecture IV., Vol. I.) asserts that—

The Loss of Potential Energy = The Gain of Kinetic Energy.

Consequently, at the instant when the monkey strikes the head of the pile, the energy of the monkey is wholly Kinetic. The work done in driving the pile into the ground is immediately derived from the kinetic energy of the falling weight, but the whole of this energy is not thus employed, for the faces of the pile and monkey have been heated by the blow. This shows that part of the energy stored in the falling weight has been transformed into heat energy. Further, at the instant of striking, a loud noise is heard, which shows that there is also a transformation into sound energy. Thus, energy appears under many different forms, such as mechanical, electrical, chemical, heat, light, sound, &c., and can, by suitable arrangements, be changed from one kind into any of the others. In nature all is change or transformation, but there is no annihilation; so what appears as a loss to the engineer simply means change into some other form which he does not desire, but which he has no power to entirely prevent.

Expression for Kinetic Energy.—We have already seen, that the expression for mechanical potential energy is:—

$$\textbf{Potential Energy} = E_P = W h$$

Where, W = Weight of body,

And, h = Height of body above zero level.

It now remains to determine the expression for kinetic energy.

First, take the case of the raised weight whose potential energy in its highest position is $W h$, and suppose it to fall freely. Its kinetic energy at the instant when it strikes the ground is:—

$$E_K = W h.$$

But, if v = velocity at that instant, we have:—

$$v^2 = 2 g h,$$

Or,
$$h = \frac{v^2}{2 g}.$$

∴
$$E_K = \frac{W v^2}{2 g} \quad \ldots\ldots\ldots \quad \text{(IV)}$$

Thus, if the monkey of the pile driver weigh 10 cwts., and reaches the pile with a velocity of 40 feet per second, it has kinetic energy :—

$$E_K = \frac{10 \times 112 \times 40^2}{2 \times 32} = 28{,}000 \text{ ft.-lbs.}$$

If the pile be driven 3 inches into the ground at each blow, what is the *mean* resistance offered to its motion, supposing there are no losses from heating, &c.?

Let R_m = Mean resistance of ground in lbs.
" s = Distance *in feet* through which the pile is driven.

Then, $\left.\begin{array}{l}\textit{Work done in driv-}\\ \quad\textit{ing the pile}\end{array}\right\} = \left\{\textit{Work given out by monkey.}\right.$

But, $\left.\begin{array}{l}\textit{Work given out by}\\ \quad\textit{monkey}\end{array}\right\} = \left\{\begin{array}{l}\textit{Kinetic energy at instant of}\\ \quad\textit{striking the pile + Work}\\ \quad\textit{done in falling through } 3''.\end{array}\right.$

$$\therefore \qquad R_m \times s = \frac{W v^2}{2g} + W \times s,$$

$$i.e., \qquad R_m \times \frac{3}{12} = 28{,}000 + 10 \times 112 \times \frac{3}{12},$$

$$\therefore \qquad R_m = 113{,}120 \text{ lbs.} = 50{\cdot}5 \text{ tons.}$$

Energy Equations.—The expression for the kinetic energy given in equation (IV) is perfectly general, and therefore independent of the manner in which the velocity, v, is acquired. That is to say, if a body of weight, W, be moving with a velocity, v, in any direction whatever, its kinetic energy is still given by the equation :—

$$E_K = \frac{W v^2}{2g}.$$

For, manifestly, the direction of motion cannot in any way affect its energy state, other things being equal. Nevertheless, we can deduce the expression from more general considerations as follows :—

Let a body of weight, W, have its velocity changed in magnitude from v_1 to v_2 by a constant force P, acting through a distance s. Then—

$\left.\begin{array}{l}\textit{Change of body's kinetic}\\ \quad\textit{energy}\end{array}\right\} = \left\{\begin{array}{l}\textit{Work done by or against}\\ \quad\textit{the force}\end{array}\right.$

And, *Work done by or on* P = P × s.

But, by eqation (I), $P = \frac{W a}{g}$,

And, by equation (VI), Lecture XX.,

$$s = \frac{v_2^2 - v_1^2}{2a},$$

$$\therefore \qquad P \times s = \frac{W(v_2^2 - v_1^2)}{2g} = \text{change of } E_K . \text{(V)}$$

If the body start from rest, and have a final velocity, v, we get:—

$$\textbf{Change of } E_K = \frac{W v^2}{2g} = \textbf{Total } E_K \textbf{ in body,}$$

which is just the same result as that given by equation (IV).

Next, take the case of a body moving with uniform velocity, v, against some constant resistance, F, which resistance may be frictional or otherwise. To maintain this constant speed a force equal to F must act on the body in opposition to the resistance; but no part of this force is employed in changing the kinetic energy of the body, since, by supposition, no change occurs in its velocity. The kinetic energy of the body is constantly $= \frac{W v^2}{2g}$, and the *Work done against resistances* = F s.

If, now, some other force, P, acts on the body in the direction of motion, the velocity will change, and, therefore, the energy of the body will also change.

Let Q = Resultant force acting on body = $P \sim F$.
„ v_1, v_2 = Velocities of body before and after action of P.
„ s = Distance through which body moves under P.

$$\text{Then,} \qquad Q \times s = F \times s + \frac{W(v_2^2 - v_1^2)}{2g} \quad . \ . \ . \ \text{(VI)}$$

This is a very general equation of energy, and is sometimes stated thus:—

Energy exerted = Work done + Change in Kinetic Energy.

EXAMPLE VI.—The height and length of an inclined plane are 20 feet and 100 feet respectively: a body weighing 100 lbs. is placed at the top of the plane and allowed to slide along its whole length; the coefficient of friction between the plane and the body is 0·15; how many units of work (foot-pounds) are accumulated in the body, and what is its velocity when it reaches the foot of the plane? (You may assume the pressure on the plane equal to the weight of the body). (S. & A. Adv. Exam.)

Answer.—Let F = Resultant force urging body down the plane.

Then, $F = W \sin \alpha - \mu R = W \sin \alpha - \mu W$, very approximately,

$$,, = 100\left(\frac{20}{100} - 0{\cdot}15\right) = 5 \text{ lbs.}$$

When body reaches the foot of the plane, we have:—

$$E_K = F \times l = 5 \times 100 = 500 \text{ ft.-lbs.}$$

Let v = Velocity at foot of plane.

Then, $$\frac{W v^2}{2g} = E_K.$$

i.e., $$\frac{100 \times v^2}{2 \times 32} = 500,$$

$\therefore$ $$v = 17{\cdot}9 \text{ ft. per sec.}$$

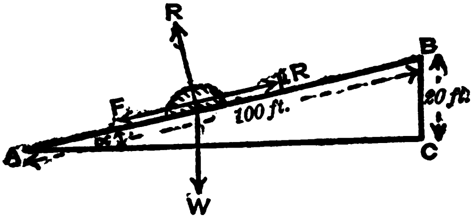


Diagram Illustrating Example VI.

The kinetic energy at the foot of the plane could be obtained immediately from equation (VI), thus:—

$$\textit{Energy exerted} = W \sin \alpha \times l = 100 \times \frac{20}{100} \times 100 = 2000 \text{ ft.-lbs.}$$

$$\textit{Work done} = \mu W l = 0{\cdot}15 \times 100 \times 100 = 1500 \text{ ft.-lbs.}$$

$$\textit{Change in } E_K = \textit{Energy at foot of plane} = \frac{W v^2}{2g}.$$

$\therefore$ **Energy exerted = Work done + Change in E_K.**

$$\therefore \qquad 2000 = 1500 + \frac{W v^2}{2 g}.$$

$$\therefore \quad \left.\begin{matrix}E_K \text{ at foot of} \\ \text{plane}\end{matrix}\right\} = \frac{W v^2}{2 g} = 500 \text{ ft.-lbs.}$$

$$\therefore \text{ also,} \qquad v = \sqrt{\frac{500 \times 2 \times 32}{100}} = 17{\cdot}9 \text{ ft. per sec.}$$

Example VII.—Show by a diagram the amount of work done in *slowly compressing* a spiral spring through 6 inches, supposing the spring to shorten 1 inch for every 100 lbs. pressure. If a weight of 100 lbs. *falls* from a height of 4 feet on the top of the spring, how much will it be compressed? (S. & A. Adv. Exam., 1892.)

Answer.—As explained in Lecture II., Vol. I., the diagram of work done in slowly compressing a spiral spring, is a right angled triangle whose base represents the compression produced, and its perpendicular side the force required to produce that compression.

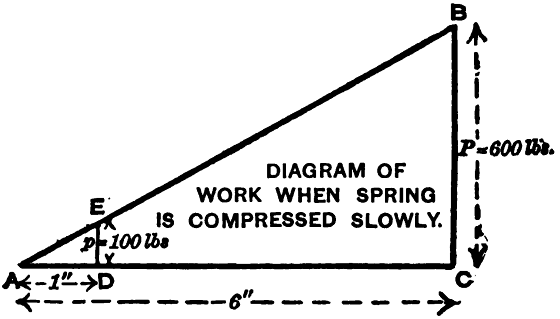


Diagram Illustrating Example VII.

Let A B C represent the diagram of work done in slowly compressing the spring.

Where, A C = Compression produced = 6".

And, C B = Force required = P.

Let, D E = Force required to compress spring 1", = p = 100 lbs.

Then, $P : p = CB : DE = AC : AD$.

That is, $P : 100 = 6'' : 1''$.

$\therefore$ $P = 600$ lbs.

$\therefore$ Work done $= \frac{1}{2} P \times L = \frac{1}{2} \times 600 \times 6'' = 1800$ in.-lbs.

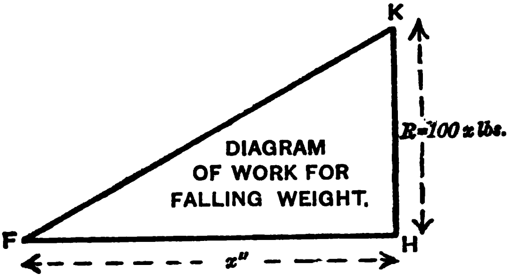


Diagram Illustrating Example VII.

Next, suppose the spring to be compressed by a weight which falls from a height of 4 feet.

Let x = Number of *inches* by which spring is compressed by falling weight.

Then, $48 + x$ = Number of inches through which weight actually falls.

Since a force of 100 lbs. is required to compress the spring 1 inch, a force $R = 100\,x$ lbs. will be required to compress it x inches.

But,

Work done in compressing spring } = { *Work done by falling weight.*

$\therefore$ $\frac{1}{2} R \times x = W(48 + x)$.

$\therefore$ $\frac{1}{2} \times 100x \times x = 100(48 + x)$.

That is, $x^2 = 2x + 96$.

Or, $x = 10{\cdot}85$ inches.

Example VIII.—A blowing-fan 30 inches in diameter revolves at a speed of 1,000 revolutions per minute, and propels the wind with a velocity equal to $\frac{7}{8}$ of the velocity of the tips of the vanes; the wind is driven through a pipe having a sectional area of 200 square inches. Neglecting the power that is required to overcome friction, will you show the amount of power which is required to force the above quantity of air? Work it out in arithmetic, either by the law of falling bodies or in any better way that may suggest itself to you. (S. & A. Hons. Exam.)

ANSWER.—This is simply a question on work and energy.

Let W = Weight of air expelled from fan *per second.*
„ A = Sectional area of delivery pipe of fan = 200 sq. ins.
„ v = Velocity of air as it leaves tips of vanes.

Then, $v = \frac{7}{8} \times$ *Velocity of tips of vanes*

$$„ = \tfrac{7}{8} \times \pi d n$$

$$„ = \frac{7}{8} \times \frac{22}{7} \times \frac{30}{12} \times \frac{1000}{60} = 114{\cdot}6 \text{ ft. per sec.}$$

$$\left.\begin{array}{l}\textit{Volume of air expelled}\\ \textit{per second}\end{array}\right\} = A v$$

$$„ \quad „ \quad = \frac{200}{144} \times 114{\cdot}6 = 159{\cdot}16 \text{ cub. ft. per sec.}$$

By calculation from the density of air, it can be shown that *13 cubic feet of air at atmospheric pressure weigh 1 lb. very nearly,* and assuming this, we get :—

$$\textit{Weight of } 159{\cdot}16 \textit{ cub. ft.} = W = \frac{159{\cdot}16}{13} = 12{\cdot}24 \text{ lbs. nearly.}$$

$$\text{Hence, } \textit{Work done per sec.} = \textit{Energy exerted} = \frac{W v^2}{2 g}$$

$$„ \quad „ \quad = \frac{12{\cdot}24 \times 114{\cdot}6^2}{2 \times 32} \text{ ft.-lbs. per sec.}$$

$$„ \quad „ \quad = \frac{12{\cdot}24 \times 114{\cdot}6^2 \times 60}{2 \times 32} \text{ ft.-lbs. per min.}$$

$$\therefore \quad \text{H.P. exerted} = \frac{12{\cdot}24 \times 114{\cdot}6^2 \times 60}{2 \times 32 \times 33{,}000} = 4{\cdot}57 \text{ nearly.}$$

Lecture XXI.—Questions.

1. Define momentum, and state how it is measured. State and explain, by aid of examples, Newton's three laws of motion. A shot weighing half a ton is fired from a 100-ton gun with a velocity of 2,000 feet per second. Neglecting the weight of the powder, find the velocity of the gun's recoil. *Ans.* 10 feet per second.

2. A man weighing 140 lbs. descends in a lift with an acceleration equal to $\frac{1}{8}g$. What pressure does he exert on the floor of the lift? How would your answer differ if the lift were ascending instead of descending? *Ans.* 122·5 lbs.; 157·5 lbs.

3. A railway train, exclusive of engine, weighs 150 tons, and in starting along a level line from rest it attains a speed of 30 miles an hour in 5 minutes. What has been the mean pull between the engine and train, the resistance being taken at 10 lbs. per ton? (S. & A. Adv. Exam., 1887.) *Ans.* 3,040 lbs.

4. A locomotive and its train weigh 220 tons, and the frictional resistance at all speeds may be taken at 2,000 lbs. If the tractive force of the engine is constantly 3,500 lbs., find in what time from starting the train can attain a speed of 40 miles per hour (1) on a level line, and (2) going down an incline of 1 in 220. Find, also, the distance travelled in both cases in attaining the above speed. *Ans.* (1) 10 minutes 2 seconds, 3·34 miles; (2) 4 minutes 1 second, 1·34 miles.

5. In a steam engine, the piston, which is 40 inches diameter and weighs 2,000 lbs., comes off the rod just as it is commencing its inward stroke. The mean steam pressure is 50 lbs. per square inch. Find the velocity with which the piston will strike the cover at the opposite end of the cylinder, the stroke being 4 feet. *Ans.* 89·7 feet per second.

6. One end of a string is fixed; it then passes over a movable pulley to which a weight, W, is attached. The string then passes over a fixed pulley, and a smaller weight, w, is attached to its other end, all three sections of the string being vertical. Show that, neglecting the weights of the pulleys, the acceleration with which W descends is $\left(\frac{W - 2w}{W + 4w}\right)g$. Verify this result (1) when w is small compared with W, and (2) when W is small compared with w. (Wool. Roy. Milly. Acad. Exams.)

7. It is very evident that a railway train requires a considerable amount of force to set it in motion, but there is a popular notion existing that a less amount of power or force is required to bring the same train to a state of rest. Will you explain clearly the natural principles upon which the whole case depends, and compare the force necessary both for giving motion to the train and in producing the opposite condition? (S. & A. Hons. Exam.)

8. Distinguish between work and energy, and between potential and kinetic energy. Give examples of both forms of energy. State the principle of the conservation of energy, and show its connection with the axiom that "perpetual motion" is impossible. A simple pendulum is pulled aside till its heavy bob is raised h inches vertically, and then let go. Find its velocity when it passes its lowest point. *Ans.* $\sqrt{\frac{gh}{6}}$.

9. Prove the formula which gives the number of units of work stored up in a given weight when moving with a given velocity. A weight of 100 lbs. is moving with a velocity of 64 feet per second, how many foot-pounds of work have been expended in producing this result? (S. & A. Adv. Exam.) *Ans.* 6,400 foot-pounds.

10. A hammer head weighs 5 tons and reaches the anvil with a velocity of 10 feet per second; what amount of energy, measured in foot-pounds, is stored up in the hammer at the instant of the blow? *Ans.* 17,500 foot-pounds.

11. The head of a steam hammer weighs 10 cwts., and has a fall of 8 feet. If it indent the iron on which it falls by 1 inch, find the mean force exerted on the iron during compression. (S. & A. Adv. Exam., 1889.) *Ans.* 970 cwts.

12. Of two steam hammers, one weighs 5 tons and reaches the anvil with a velocity of 10 feet per second, and the other weighs 10 tons and reaches the anvil at a velocity of 5 feet per second; will you compare and distinctly characterise the conditions of the blow of each of the two hammers? (S. & A. Hons. Exam.)

13. Referring to a steam hammer, in which steam is admitted above the piston to assist gravitation, will you describe the combination of forces at work in producing the blow, and, as far as you may be able, the nature of the blow as depending on velocity and mass or weight of the hammer at the moment of impact? (S. & A. Hons. Exam.)

14. What do you understand by energy, and how is it measured? The head of a steam hammer weighs 50 cwts., steam is admitted on the under side for lifting only, and there is a drop of 5 feet. What will be the average compressive force exerted during a blow from this hammer, on the supposition that the duration of the blow—that is, the time during which the hammer is compressing the iron under operation—is $\frac{1}{80}$ second? (S. & A. Hons. Exam., 1887.) *Ans.* 2,236 cwts.

15. The monkey of a pile driver weighs 15 cwts., and the drop is 6 feet. The blow causes the pile to go down through 4 inches; what is the frictional resistance of the earth? (S. & A. Hons. Exam., 1881.) *Ans.* 285 cwts.

16. Compare the force expended in pile driving by a ram or monkey of 1 ton falling 20 feet, with that of a weight of 2 tons falling 10 feet. If one blow of the former moves the pile 9 inches, what is the average resistance that is opposed to its motion? (S. & A. Adv. Exam., 1896.) *Ans.* (1) 2 : 1, (2) 27·7 tons.

17. Two bodies, weighing 5 lbs. and 3 lbs. respectively, are connected by a perfectly flexible weightless string which passes over a smooth pulley. The heavier body draws up the lighter. When it has fallen through 5 feet, what is the kinetic energy of the bodies and the velocity? ($g = 32.$) (S. & A. Theor. Mechs. Elem. Exam., 1880.) Determine also the acceleration of the system, and the tension in the string. *Ans.* (1) 10 foot-pounds, (2) 8·94 feet per second; (3) 8 feet per second per second; (4) 3·75 lbs.

18. State Newton's third law of motion, and give his illustrations of it. Weights of 5 and 11 lbs. are connected by a weightless thread. The latter is placed on a smooth horizontal table, while the former hangs over the edge. If the bodies are then allowed to move under the action of gravity, what is the tension of the thread? (S. & A. Theor. Mechs. Adv. Exam., 1876.) Find, also, the acceleration produced, and the kinetic energy of the system at the end of 4 seconds. *Ans.* (1) 3·44 lbs., (2) 10 feet per second per second, (3) 400 foot-pounds.

19. A train of locomotive and carriages weighs 60 tons. If it be sup-

posed to run down an incline of 1 in 265 for 7½ miles, starting with zero velocity, unopposed by anything but its own inertia, and unaccelerated by anything but its own weight; what would be its velocity, its momentum, and its kinetic energy at the foot of the incline? (C. & G. Mech. Eng. Hons. Exam., 1884.) *Ans.* 97·84 feet per second, 5,870 foot-tons per second, 8,925 foot-tons.

20. What meaning do you attach to the phrase *horse-power?* A fire-engine pump is provided with a nozzle, the sectional area of which is 1 square inch, and the water is projected through the nozzle with a velocity of 130 feet per second; find the horse-power of the engine required to drive the pump, irrespective of the loss by resistance of the working parts. The weight of a cubic foot of water is 62½ lbs. (S. & A. Hons. Exam.) *Ans.* 27·1 H.P.

21. State Newton's second law of motion. Explain briefly how the measure of force is derived from this law. In the equation $P = mf$, in what units is P, when the units of mass, distance, and time are a pound, a foot, and a second? (S. & A. Adv. Theor. Mechs. Exam., 1896.)

22. A steam engine is employed to raise coals, and it is calculated that in order to set in motion the winding drums, flywheel, cages, ropes, &c., which are concerned in the motion, it has to do the work of imparting a linear velocity of 36 feet per second, to 60 tons of material in *half-a-minute* at each lift. What effective horse-power is consumed in overcoming the inertia of the aggregate weight of 60 tons, and in setting up the velocity, estimated as above stated, in the time assigned? (S. & A. Hons. Exam.) *Ans.* 165 H.P.

23. In lifting water into a tender by a scoop running along a trough while the train is going at rapid speed, the height of the lift is 7½ feet. What speed in miles per hour will just cause the water to be lifted through that height? (S. & A. Hons. Exam.) *Ans.* 15 miles per hour nearly.

24. A vertical pipe, carried by the tender of a locomotive engine, and terminating in a scoop with a flat mouth, picks up water from a trough laid on a railway. If the speed of the engine and tender be 22 miles per hour, find the height to which the water will rise in the pipe. Upon what theory do you proceed? (S. & A. Hons. Exam., 1882.) *Ans.* 16·3 feet.

LECTURE XXII.

CONTENTS.—Energy of a Rotating Body—Moment of Inertia of a Body about an Axis—Definitions of Moment of Inertia and Radius of Gyration—Propositions I., II., and III.—Methods of Calculating Moments of Inertia—Examples I., II.; and III.—Tables of Radii of Gyration of Solids and Sections—Equation of Energy for a Rotating Body—Examples IV., V., VI., and VII.—Determination of Energy of Flywheels—Centripetal and Centrifugal Forces—Definitions of Centripetal and Centrifugal Forces—Example VIII.—Straining Actions due to Centrifugal Forces—Example IX.—Questions.

Energy of a Rotating Body.—The deduction of the equation for energy of rotation is complicated by the fact that particles of the body at different distances from the axis of rotation possess different energies, due to their different linear velocities. To obtain the energy of the whole body, we must, therefore, take the sum of the energies of the various particles composing it. In general, this process must be performed by the aid of higher mathematics; and even then, only in those cases in which the bodies are of regular geometrical form.

Moment of Inertia of a Body about an Axis.—Before deducing the expression for the kinetic energy of a rotating body, it may be as well to explain certain terms and quantities which we shall have occasion to make use of.

DEFINITION.—**If the mass of every particle of a body be multiplied by the square of its distance from a given axis, the sum of the products is called the Moment of Inertia of the body about that axis.**

Let I = Moment of inertia of the body about a given axis.
„ m = Mass of any particle or element of body.
„ r = Distance of m from the given axis.

Then,
$$I = \Sigma m r^2. \quad \ldots \ldots \quad \text{(I)}$$

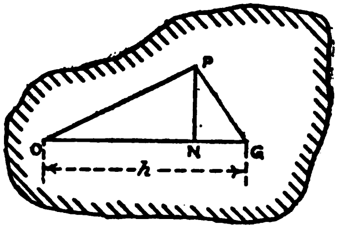
DEFINITION.—**If M be the mass of a body, and k be such a quantity that Mk^2 is its Moment of Inertia about a given axis, then k is called the Radius of Gyration of the body about that axis.**

$$\left.\begin{aligned} \text{Thus,} \quad M k^2 &= I \\ \text{Or,} \quad k^2 &= \frac{I}{M} \end{aligned}\right\} \quad \ldots \ldots \quad \text{(II)}$$

The following PROPOSITIONS relating to moments of inertia and radii of gyration are so important that we here give their proofs in full:—

PROPOSITION I.—If M be the mass of any body, I_G the moment of inertia about any axis through its centre of gravity, G, and I_O, that about a parallel axis through any other point, O, at a distance, *h*, from G, then:—

$$I_O = I_G + M h^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(III)}$$



MOMENTS OF INERTIA ABOUT PARALLEL AXES.

Let G, and O, be the points of intersection of the axes with the plane of the paper, which is at right angles to them. Let P be any particle of mass, *m*. Draw P N perpendicular to O G.

Then, in triangle O P G, we get (Euc. II., 12):—

$$OP^2 = PG^2 + OG^2 - 2\,OG \cdot GN.$$

Multiplying both sides by *m* the mass of particle at P, we get:—

$$m \cdot OP^2 = m \cdot PG^2 + m \cdot OG^2 - 2m \cdot OG \cdot GN.$$

Repeating this process for every other particle of the body, and adding the results, we have:—

$$\Sigma m \cdot OP^2 = \Sigma m \cdot PG^2 + \Sigma m \cdot OG^2 - 2\Sigma m \cdot OG \cdot GN.$$

But, clearly, $\Sigma m \cdot OP^2 = I_O$, and $\Sigma m \cdot PG = I_G$.

And, since $OG = h = \text{constant},$

$\therefore$ $\Sigma m \cdot OG^2 = OG^2 \Sigma m = h^2 M$, or $M h^2$.

Also, $2\Sigma m \cdot OG \cdot GN = 2\,OG \cdot \Sigma m \cdot GN = 2h\Sigma m \cdot GN.$

But the quantity, $\Sigma m \cdot GN$, is the sum of the moments of the various particles about their centre of gravity, G, and is therefore zero, from the definition of the centre of gravity.

$$\therefore \qquad 2\Sigma m \cdot \mathrm{O\,G} \cdot \mathrm{G\,N} = 2h\Sigma m \cdot \mathrm{G\,N} = 0.$$

$$\therefore \qquad \mathrm{I}_0 = \mathrm{I}_G + \mathrm{M}\,h^2.$$

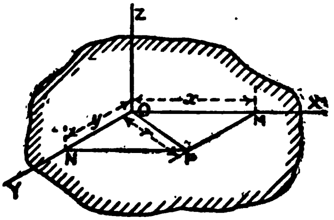
Corollary.—Denoting the radii of gyration of the body about the axes, O, and G, by k_0, and k_G, respectively, we get:—

$$\mathrm{M}\,k_0^2 = \mathrm{M}\,k_G^2 + \mathrm{M}\,h^2.$$

$$\therefore \qquad k_0^2 = k_G^2 + h^2. \quad . \; . \; . \; . \; . \; . \; . \quad \text{(IV)}$$

Proposition II.—If I_x and I_y respectively denote the moments of inertia of a lamina, or plane area, about two axes 0 X, 0 Y, at right angles, lying in the plane of the lamina or area, and I_z, the moment of inertia about an axis, 0 Z, through 0, perpendicular to the plane of the lamina or area; then I_z is equal to the sum of I_x, and I_y.

i.e., $$\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y. \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(V)}$$



Moment of Inertia of Lamina about Rectangular Axes.

Take any particle, P, of mass, m, and draw P M, and P N, perpendicular to O X and O Y respectively.

Let x and y denote the co-ordinates of P, with respect to the axes, O X, O Y, so that O M $= x$, O N $= y$, and O P $= r$. Then:—

Moment of inertia of P about O X $= m \cdot \mathrm{P\,M}^2 = m \cdot y^2$.

„ „ O Y $= m \cdot \mathrm{P\,N}^2 = m \cdot x^2$.

„ „ O Z $= m \cdot \mathrm{O\,P}^2 = m \cdot r^2$.

But, $$r^2 = y^2 + x^2,$$

$$\therefore \qquad m \cdot r^2 = m \cdot y^2 + m \cdot x^2.$$

Hence, the moment of inertia of P, about the axis, O Z, is equal to the sum of the moments of inertia of the same particle about the axes, O X, and O Y. But this is equally true for every other particle.

$$\therefore \qquad \Sigma m r^2 = \Sigma m y^2 + \Sigma m x^2.$$

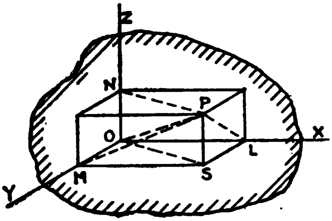
$$i.e., \qquad I_z = I_x + I_y.$$

COROLLARY.—Denoting the radii of gyration of the lamina, about the axes by the letters k_x, k_y, k_z, we get:—

$$k_z^2 = k_x^2 + k_y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{VI})$$

PROPOSITION III.—If I_x, I_y, I_z respectively denote the moments of inertia of any body about three rectangular axes drawn from any point, O, in the body, then the sum, $I_x + I_y + I_z$, is equal to twice the moment of inertia, I_0, of the body about the point, O.

$$i.e., \qquad I_x + I_y + I_z = 2\,I_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{VII})$$



MOMENT OF INERTIA OF BODY ABOUT RECTANGULAR AXES.

Let O X, O Y, O Z be the three rectangular axes drawn from any point, O; P, any particle of mass m, whose co-ordinates are x, y, z, so that $OL = x$, $OM = y$, $ON = z$, and $OP = r$. Then:—

Moment of inertia of P about $OX = m \cdot PL^2 = m(y^2 + z^2)$.

" " $OY = m \cdot PM^2 = m(z^2 + x^2)$.

" " $OZ = m \cdot PN^2 = m(x^2 + y^2)$.

$$\therefore \qquad I_x = \Sigma m(y^2 + z^2) = \Sigma m y^2 + \Sigma m z^2,$$

$$I_y = \Sigma m(z^2 + x^2) = \Sigma m z^2 + \Sigma m x^2,$$

$$\text{And,} \qquad I_z = \Sigma m(x^2 + y^2) = \Sigma m x^2 + \Sigma m y^2.$$

$$\therefore \qquad I_x + I_y + I_z = 2\,\{\Sigma m x^2 + \Sigma m y^2 + \Sigma m z^2\}$$

$$\text{"} \quad \text{"} \qquad = 2\,\Sigma m(x^2 + y^2 + z^2),$$

$$i.e., \qquad I_x + I_y + I_z = 2\,\Sigma m r^2 = 2\,I_0.$$

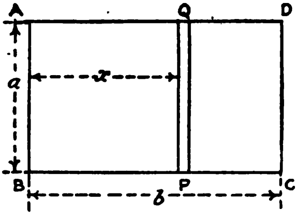
Corollary.—Denoting the radii of gyration of the body about the three axes and the point, O, by the letters k_x, k_y, k_z, and k_o respectively, we get:—

$$k_x^2 + k_y^2 + k_z^2 = 2\,k_o^2. \quad . \quad . \quad . \quad . \quad \text{(VIII)}$$

Proposition II. is a particular case of this general one. There O P and O S will always coincide, and, therefore, I_z and I_o are identical. By putting $I_o = I_z$ in equation (VII), we at once get equation (V).

Methods of Calculating Moments of Inertia.—We shall show by working out a few examples how the moments of inertia or radii of gyration can be calculated in certain cases, wherein the density is uniform.

Example I.—Determine the moment of inertia and radius of gyration of a rectangular lamina (1) about its shorter edge, (2) about an axis in its plane through its *c.g.* and parallel to a short edge, and (3) about an axis through its *c.g.* perpendicular to its plane.



Moment of Inertia of Rectangle about AB.

Answer.—Let A B C D be the rectangular lamina, and let the edge A B = a, and B C = b.

(1) *About the shorter edge* A B. Divide the rectangle into n, equal and narrow strips, P Q, parallel to the axis A B.

Let M = Mass of whole figure, A B C D.
„ m = Mass of elementary rectangle, P Q.
„ x = Distance of P Q from axis A B.
„ h = Breadth of elementary strip P Q $= \frac{b}{n}$.

The whole of the strip P Q is at the same distance from A B,

$$\therefore \quad \left.\begin{matrix}\textit{Mom. of inertia of}\\ \textit{element } \text{P Q}\end{matrix}\right\} = m\,x^2.$$

$$\therefore \quad \left.\begin{matrix}\textit{Mom. of inertia of}\\ \textit{whole figure}\end{matrix}\right\} = \Sigma\, m\,x^2.$$

But
$$m : \text{M} = h : b.$$

$$\therefore \quad m = \frac{\text{M}\,h}{b}.$$

$$\therefore \quad \text{I} = \frac{\text{M}}{b}\Sigma\,h\,x^2. \quad . \quad . \quad . \quad . \quad (1)$$

Beginning at edge A B, the distances of the various strips from this edge will be $x_0 = 0$, $x_1 = h$, $x_2 = 2h$, . . . $x_n = nh$.

$$\therefore \quad \Sigma h x^2 = h(0^2 + h^2 + 2^2 h^2 + 3^2 h^2 + \ldots + n^2 h^2),$$

$$,, \quad = h^3(1^2 + 2^2 + 3^2 + \ldots + n^2),$$

$$,, \quad = h^3 \frac{n(n+1)(2n+1)}{6}, \text{ [See Treatises on Algebra]}$$

$$,, \quad = \frac{(nh)^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

When the number of strips, n, is infinitely large, the reciprocal, $\frac{1}{n}$, will be infinitely small, and may, therefore, be neglected.

Also, $$(nh)^3 = b^3.$$

$$\therefore \quad \Sigma h x^2 = \frac{b^3}{6} \times 2 = \tfrac{1}{3} b^3,$$

$\therefore$ From eqn. (1), $$\text{I} = \tfrac{1}{3}\text{M} b^2.$$

Let k = Radius of gyration, then :—

$$\text{M} k^2 = \text{I},$$

$$\therefore \quad k^2 = \tfrac{1}{3} b^2,$$

Or, $$k = \frac{b}{\sqrt{3}}.$$

[The above method of finding the moment of inertia is precisely the same as that employed in higher mathematics. For those who understand the calculus we here repeat the above calculation, using its notation.

Let dx = Breadth of elementary strip, P Q.

Then, $$m = \frac{\text{M}}{b} dx.$$

$$\therefore \quad d\text{I} = \frac{\text{M}}{b} x^2 dx.$$

$$\therefore \quad \text{I} = \frac{\text{M}}{b}\int_0^b x^2 dx = \frac{\text{M}}{b}\left[\frac{x^3}{3}\right]_0^b = \frac{\text{M}}{b} \cdot \frac{b^3}{3} = \tfrac{1}{3}\text{M} b^2.]$$

(2) *About an axis through c.g. parallel to edge* A B.—We may obtain the moment of inertia in this case by proceeding in

exactly the same way as before, but there is no need for this repetition, as we can very easily get the result from the relation given in PROPOSITION I.

Let I_G = Moment of inertia of the rectangle about an axis through its *c.g.* parallel to the edge A B.
" I = Moment of inertia about the edge A B = $\frac{1}{3} M b^2$.
" h = Distance between these axes = $\frac{1}{2} b$.

Then, from equation (III) :—

$$I = I_G + M h^2.$$

$$\therefore \quad I_G = I - M h^2 = \tfrac{1}{3} M b^2 - \tfrac{1}{4} M b^2 = \tfrac{1}{12} M b^2.$$

Also, $$k_G^2 = \tfrac{1}{12} b^2.$$

(3) *About an axis through c.g. perpendicular to plane.*

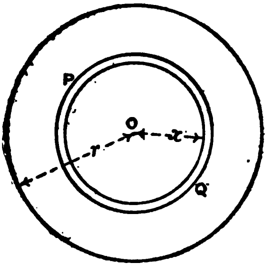
$$\left.\begin{array}{l}\textit{Moment of inertia about an axis}\\ \textit{through c.g. parallel to } \mathrm{A\,B}\end{array}\right\} = I_x = \tfrac{1}{12} M b^2.$$

$$\left.\begin{array}{l}\textit{Moment of inertia about an axis}\\ \textit{through c.g. parallel to } \mathrm{B\,C}\end{array}\right\} = I_y = \tfrac{1}{12} M a^2$$

But, from equation (V), $$I_z = I_x + I_y.$$

$$\therefore \left.\begin{array}{l}\text{Moment of inertia about an axis}\\ \text{through } c.g. \text{ perpendicular to}\\ \text{the plane}\end{array}\right\} = I_z = \tfrac{1}{12} M (a^2 + b^2).$$

EXAMPLE II.—Determine the radius of gyration of a circular disc about an axis through its centre perpendicular to its plane.



RADIUS OF GYRATION OF A CIRCULAR DISC.

ANSWER.—Divide the disc into n, equal, narrow rings of breadth, h. Taking one of these rings, P Q, let x be its distance from the centre, O.

Let M = Mass of the disc.
" m = Mass of the elementary ring, P Q.
" r = Radius of the disc.

Then, $m : M$ = *area of ring* : *area of disc.*

i.e., $m : M = 2 \pi x h : \pi r^2$.

$$\therefore \quad m = \frac{2 M}{r^2} x h.$$

The whole of the elementary ring is at the same distance from the centre :—

$$\therefore \left.\begin{matrix}\textit{Moment of inertia of elementary}\\ \textit{ring about the axis through O}\end{matrix}\right\} = m\,x^2 = \frac{2\,\mathrm{M}}{r^2}\,x^3\,h.$$

$$\therefore \left.\begin{matrix}\textit{Moment of inertia of the whole}\\ \textit{disc}\end{matrix}\right\} = \mathrm{I} = \frac{2\,\mathrm{M}}{r^2}\,\Sigma\,x^3\,h. \qquad (1)$$

Beginning at the centre, O, the distances of the various rings will be $x_0 = 0, x_1 = h, x_2 = 2h, \ldots x_n = nh.$

$$\therefore \quad \Sigma\,x^3\,h = (0^3 + 1^3\,h^3 + 2^3\,h^3 + \ldots + n^3\,h^3)\,h$$

$$\text{,,} \quad = (1^3 + 2^3 + 3^3 + \ldots + n^3)\,h^4$$

$$\text{,,} \quad = \tfrac{1}{4}\,n^2\,(n + 1)^2\,h^4 \qquad \text{[See Treatises on Algebra]}$$

$$\text{,,} \quad = \frac{(n\,h)^4}{4}\cdot\left(1 + \frac{1}{n}\right)^2.$$

When n is infinitely great, the reciprocal, $\frac{1}{n}$, will be infinitely small, and may be neglected.

Also, $(n\,h)^4 = r^4.$

$$\therefore \quad \text{From equation (1),} \quad \mathrm{I} = \frac{2\,\mathrm{M}}{r^2} \times \frac{r^4}{4} = \tfrac{1}{2}\,\mathrm{M}\,r^2.$$

$$\therefore \quad k^2 = \tfrac{1}{2}\,r^2. \quad \ldots\ldots\ldots \quad (2)$$

[These results may also be obtained by the aid of the calculus, thus :—

Let $d\,x$ = Breadth of elementary ring.

Then, $m = \frac{2\,\mathrm{M}}{r^2}\,x\,d\,x.$

$$\therefore \quad d\,\mathrm{I} = \frac{2\,\mathrm{M}}{r^2}\,x^3\,d\,x.$$

$$\therefore \quad \mathrm{I} = \frac{2\,\mathrm{M}}{r^2}\int_0^r x^3\,d\,x = \frac{2\,\mathrm{M}}{r^2}\cdot\frac{r^4}{4} = \tfrac{1}{2}\,\mathrm{M}\,r^2.$$

If, however, the disc be annular, the outside and inside radii being R and r respectively, we get :—

$$m = \frac{2\,\mathrm{M}}{\mathrm{R}^2 - r^2}\cdot x\,d\,x.$$

And,
$$I = \frac{2M}{R^2 - r^2}\int_r^R x^3\,dx$$

$$,, = \frac{2M}{R^2 - r^2} \times \frac{R^4 - r^4}{4} = \tfrac{1}{2}M(R^2 + r^2).$$

$$\therefore \qquad k^2 = \tfrac{1}{2}(R^2 + r^2).]$$

These results also express the moments of inertia and radii of gyration of a solid and of a hollow cylinder about their axes. For a cylinder can be conceived as made up of a great number of circular discs threaded together on the same axis, and the moment of inertia will just be the sum of the moments of inertia of all the discs; since the radius of gyration of each disc is independent of the thickness of the disc, it follows that the radius of gyration of the whole cylinder will be the same as that of one of the discs.

Having found the radius of gyration of a circular disc about an axis through its centre at right angles to its plane, we can very easily find its radius of gyration about a diameter.

Let k_x, k_y = Radii of gyration of disc about two diameters at right angles to each other.

,, k_z = Radius of gyration about axis through centre and perpendicular to plane.

Then,
$$k_x = k_y = k$$

And, from (2)
$$k_z^2 = \tfrac{1}{2}r^2.$$

But, from equation (VI), Proposition II., we get:—

$$k_x^2 + k_y^2 = k_z^2.$$

$$\therefore \qquad 2k^2 = k_z^2 = \tfrac{1}{2}r^2.$$

$$\therefore \qquad k^2 = \tfrac{1}{4}r^2, \text{ or } k = \frac{r}{2}.$$

If the disc be annular and of radii R and r, then the radius of gyration about any diameter, is given by the equation:—

$$k^2 = \tfrac{1}{4}(R^2 + r^2).$$

Example III.—Determine the radius of gyration of a sphere about a diameter.

Answer.—The results of Proposition III. tell us that if three mutually perpendicular axes be drawn from any point in a body, the sum of the moments of inertia of the body about these axes is equal to twice the moment of inertia of the body about that point. Suppose, then, that the point selected be the centre of the sphere, the axes will then be three mutually perpendicular diameters. But the moments of inertia about all diameters must be the same. Therefore, if I denote the moment of inertia of the sphere about any diameter and I_0 that about the centre, O, we get, from equation (VII):—

$$3\,I = 2\,I_0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

It only remains now to find the value of I_0 or $\Sigma\, m\, x^2$.

Suppose the sphere divided into a large number n, of concentric shells, the thickness of each shell being h.

Let x = Distance of any one shell from centre of sphere.
" r = Radius of sphere.
" m = Mass of shell.
" M = Mass of sphere.

Then, $\quad m : M = \text{vol. of shell} : \text{vol. of sphere},$

i.e., $\quad m : M = 4\,\pi\, x^2 h : \frac{4}{3}\pi\, r^3,$

$\therefore \quad m = \frac{3\,M}{r^3}\, x^2\, h,$

And, $\quad I_0 = \Sigma\, m\, x^2 = \frac{3\,M}{r^3}\,\Sigma\, x^4\, h.$

Beginning at the centre of the sphere and putting successively, $x_0 = 0$, $x_1 = h$, $x_2 = 2\,h$, . . . $x_n = n\,h$, we get:—

$$\Sigma\, x^4\, h = (1^4 + 2^4 + 3^4 + \ldots n^4)h^5,$$

$$\text{"} \quad = \left\{\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right\} h^5,$$

$$\text{"} \quad = \left\{\frac{1}{5} + \frac{1}{2\,n} + \frac{1}{3\,n^2} - \frac{1}{30\,n^4}\right\} n^5\, h^5.$$

When n is infinitely great, the quantity inside the brackets reduces to $\frac{1}{5}$, and

$$n^5\, h^5 = r^5.$$

$$\therefore \quad \Sigma\, x^4\, h = \tfrac{1}{5}\, r^5.$$

$$\therefore \quad I_0 = \frac{3\,M}{r^3} \times \tfrac{1}{5}\, r^5 = \tfrac{3}{5}\, M\, r^2.$$

$\therefore$ From equation (I) $I = \frac{2}{3} I_0 = \frac{2}{5} M r^2$.

Hence, $k^2 = \frac{2}{5} r^2$.

[The same result can be easily arrived at by aid of the Calculus. With the usual notation, we get :—

$$m = \frac{3\,M}{r^3} x^2\,d\,x,$$

$$\therefore \quad I_0 = \Sigma\, m\, x^2 = \frac{3\,M}{r^3}\int_0^r x^4\,d\,x = \frac{3\,M}{r^3} \times \frac{r^5}{5} = \frac{3}{5} M\,r^2,$$

$$\therefore \quad I = \frac{2}{3} I_0 = \frac{2}{5} M r^2.]$$

If the sphere be hollow, the inside radius being r, and the outside radius R, it can easily be proved that :—

$$I = \frac{2}{5} M \frac{R^5 - r^5}{R^3 - r^3}.$$

The term "moment of inertia" has been defined above with respect to a solid body only, but it is easy to see that by a slight alteration in the wording of the definition it may be made to apply equally to an area or a *section* of a solid. Accordingly, we find the terms "moment of inertia" and "radius of gyration" applied to areas as well as to solids. Thus, we speak about the moment of inertia and radius of gyration of a circle about a diameter, a triangle about its base, and so on.

We may here remark that the moment of inertia of a solid, or section of a solid, about a given axis, is always proportional to the mass of the solid, or to the area of the section as the case may be.

The following rule has been stated by Routh and will be found useful for finding the moments of inertia about an axis of symmetry :—

Moment of Inertia = *Mass* × (*sum of the squares of the perpendicular semi-axes*) ÷ (3, 4, *or* 5, *according as the body is rectangular*, *elliptical*, *or ellipsoidal*).

For the sake of reference, we here give tables of the squares of the radii of gyration for some of the more important cases of both solids and sections.

In every case the axis is taken as passing through the centre of mass of the solid or centre of area of the section, so that if the moment of inertia or radius of gyration be required about any other axis, this can easily be computed from the results given in PROPOSITIONS I., II., and III.

TABLE I.—Squares of Radii of Gyration of Solids.

	Name of Solid, and Dimensions.	Position of Axis through c.g.	Square of Radius of Gyration. $k^2 = \frac{I}{M}$
I.	Circular hoop of thin wire—Radius, r	Perp. to plane of circle	r^2
II.	Circular hoop of thin wire—Radius, r	About a diameter	$\frac{1}{2} r^2$
III.	Uniform circular rod—Length, l; radius, r	Perp. to length	$\frac{1}{12} l^2 + \frac{1}{4} r^2$
IV.	Solid circular cylinder—Radius, r	About its own axis	$\frac{1}{2} r^2$
V.	Hollow circular cylinder or ring—Radii, R, r	About its own axis	$\frac{1}{2} (R^2 + r^2)$
VI.	Thin cylindrical shell—Radius, r	About its own axis	r^2
VII.	Solid sphere—Radius, r	About a diameter	$\frac{2}{5} r^2$
VIII.	Hollow sphere—Radii, R, r	About a diameter	$\frac{2}{5} \frac{R^5 - r^5}{R^3 - r^3}$
IX.	Thin spherical shell—Radius, r	About a diameter	$\frac{2}{3} r^2$
X.	Solid cone—Radius of base, r	About its own axis	$\frac{3}{10} r^2$

TABLE II.—Squares of Radii of Gyration of Lamina and Surfaces or Sections.

	Form of Lamina, Surface, or Section.	Position of Axis through c.g.	Square of Radius of Gyration. $k^2 = \frac{I}{A}$
I.	Rectangle—Sides, a, b	Parallel to side, b	$\frac{1}{12} a^2$
II.	Rectangle—Sides, a, b	Perp. to plane of figure	$\frac{1}{12} (a^2 + b^2)$
III.	Hollow rectangle—Sides, A, B, and a, b	Parallel to sides, B, b	$\frac{1}{12} \frac{A^3B - a^3b}{AB - ab}$
IV.	Triangle — Altitude, a; base, b	Parallel to base, b	$\frac{1}{18} a^2$
V.	Circular section—Radius, r	Perp. to plane of figure	$\frac{1}{2} r^2$
VI.	Circular section—Radius, r	About a diameter	$\frac{1}{4} r^2$
VII.	Hollow circular section—Radii, R, r	Perp. to plane of figure	$\frac{1}{2} (R^2 + r^2)$
VIII.	Hollow circular section—Radii, R, r	About a diameter	$\frac{1}{4} (R^2 + r^2)$
IX.	Elliptical section—Axes, a, b	About axis, b	$\frac{1}{16} a^2$
X.	Elliptical section—Axes, a, b	Perp. to plane of figure	$\frac{1}{16} (a^2 + b^2)$
XI.	Hollow elliptical section—Axes, A, B, and a, b	About axis, B, b	$\frac{1}{16} \frac{A^3B - a^3b}{AB - ab}$

Equation of Energy for a Rotating Body.—We shall now determine the energy possessed by a rotating body.

Let W = Weight of body, and M its mass.
„ w = Weight of any particle at a distance, r, from the axis of rotation, and m its mass.
„ ω = Angular velocity of body about given axis.
„ v = Linear velocity of the particle = ωr.
„ k = Radius of gyration about the given axis.

Then, the kinetic energy of the particle $= \frac{w v^2}{2g} = \frac{w \omega^2 r^2}{2g}$.

Repeating this process for every particle composing the body, and adding the results together, we get:—

$$\left.\begin{array}{r}\text{The kinetic energy of}\\ \text{the whole body, } E_K\end{array}\right\} = \Sigma \frac{w \omega^2 r^2}{2g} = \frac{\omega^2}{2g} \Sigma w r^2 = \frac{\omega^2}{2} \Sigma m r^2,$$

since $w = m g$, and ω is the same for every particle.

But, $$\Sigma m r^2 = I = Mk^2 = \frac{Wk^2}{g}, \text{ about the given axis,}$$

$$\therefore \qquad E_K = \tfrac{1}{2} I \omega^2 = \frac{W \omega^2 k^2}{2g} \quad . \quad . \quad . \quad . \quad (IX)^*$$

Thus, the equation for the energy of a body rotating about a fixed axis is similar in form to that for a body moving without rotation.

Engineers usually measure the angular velocity of a rotating body by the number of revolutions made in unit time.

Then, if n be the number of revolutions per unit time,

$$\omega = 2 \pi n$$

$$\therefore \qquad E_K = \frac{W \times 4 \pi^2 n^2 k^2}{2g} = \frac{2 \pi^2 n^2 W k^2}{g} \quad . \quad . \quad . \quad . \quad (X)$$

We may also show, as in the previous Lecture, that, if the angular velocity changes from ω_1 to ω_2, or from n_1 to n_2 revolutions per second, then:—

---

[*] If W be expressed in absolute units or poundals, the kinetic energy will also be given in absolute units or foot-poundals; but if W be in pounds weight or in gravitation units, then the kinetic energy will be in foot-pounds.

The student should note that the *pound* is the absolute unit of mass, and, therefore, those of the above equations which contain M instead of $\frac{W}{g}$ *always* give the kinetic energy in absolute units or foot-poundals.

$$\left.\begin{array}{ll}\text{The change of kinetic energy} & = \dfrac{W(\omega_2^2 - \omega_1^2)k^2}{2g} \\ \text{Or, ,, ,,} & = \dfrac{2\pi^2 W(n_2^2 - n_1^2)k^2}{g}\end{array}\right\} \quad . \quad . \quad \text{(XI)}$$

Again, if the centre of gravity of the body be moving with a linear velocity, v, and if at the same time the body be rotating about an axis through its centre of gravity with an angular velocity, ω, then the total kinetic energy possessed by the body is:—

$$E_K = \frac{W v^2}{2g} + \frac{W \omega^2 k^2}{2g} = \frac{W}{2g}\{v^2 + \omega^2 k^2\}. \quad . \quad \text{(XII)}$$

Or, if the linear velocity changes from v_1 to v_2, while the angular velocity changes from ω_1 to ω_2, then the total change in the kinetic energy of the body during that period is:—

$$\frac{W(v_2^2 - v_1^2)}{2g} + \frac{W(\omega_2^2 - \omega_1^2)k^2}{2g} = \frac{W}{2g}\{(v_2^2 - v_1^2) + (\omega_2^2 - \omega_1^2)k^2\}. \quad \text{(XIII)}$$

EXAMPLE IV.—Sketch and describe the action of a fly-press as used for punching holes in metal plates. The balls weigh 60 lbs. each, and are fixed at a radius of 30 inches from the axis of the screw. The screw is double threaded, and of 1 inch pitch. Find what diameter of hole can be punched in a wrought-iron plate $\frac{3}{8}$ inch thick, if the strength of the plate in shear be taken at 22 tons per square inch, the resistance to shearing be overcome in the first $\frac{1}{16}$ inch, and if the balls at the instant when the punch touches the plate are moving at the rate of 60 revolutions per minute.

ANSWER.—For a sketch and description of a fly-press, the student may refer to Lecture XXI., of the Author's *Elementary Manual on Applied Mechanics.*

Let W = Weight of *each* ball = 60 lbs.

,, k = Radius of gyration of the system = $2\frac{1}{2}$ feet.

,, n = Number of revolutions *per second* = 1.

,, R = Resistance, in lbs., offered by the metal to the punch.

,, s = Distance through which R is overcome = $\frac{1}{16}$ inch $= \dfrac{1}{12 \times 16}$ feet.

,, t = Thickness of plate punched = $\frac{3}{8}$ inch.

,, d = Diameter, in inches, of the hole.

,, f = Resistance of metal to shearing = 22×2240 lbs. per square inch.

Then, $$\textit{Area sheared} = \left\{ \begin{array}{l} \textit{Area of cylindrical surface of} \\ \quad \textit{hole} = \pi\, d\, t. \end{array} \right.$$

$$\therefore \quad \left. \begin{array}{l} \textit{Mean resistance offered} \\ \quad \textit{to shearing} \end{array} \right\} = \mathrm{R} = \pi\, d\, t f.$$

$$\therefore \quad \textit{Work done against } \mathrm{R} = \mathrm{R} \times s = \pi\, d\, t f \times s.$$

But, $$\textit{Work done against } \mathrm{R} = \left\{ \begin{array}{l} \textit{Energy of moving balls at the} \\ \quad \textit{instant when punch strikes} \\ \quad \textit{metal} \end{array} \right.$$

$$,, \qquad ,, \qquad = \frac{2\,\mathrm{W} v^2}{2 g} = \frac{\mathrm{W} \times 4\,\pi^2 n^2 k^2}{g}.$$

$$\therefore \qquad \pi\, d\, t f \times s = \frac{\mathrm{W} \times 4\,\pi^2 n^2 k^2}{g}.$$

This is the general equation connecting together the given and the required quantities. By substituting the given data, and cancelling π from both sides of the equation, we get :—

$$d \times \tfrac{3}{8} \times 22 \times 2240 \times \frac{1}{12 \times 16} = \frac{60 \times 4 \times \frac{22}{7} \times 1 \times 1 \times 2\tfrac{1}{2} \times 2\tfrac{1}{2}}{32}$$

$$\therefore \qquad d = 1{\cdot}53 \text{ inch.}$$

Example V.—A flywheel weighing 4 tons is keyed to a shaft of 9 inches diameter at the journals. The radius of gyration of the wheel is 5¼ feet. At a given instant the wheel is found to be making 80 revolutions per minute, and is not acted on by any other retarding forces than the friction at its journals. Find (1) the reduction in speed after the wheel has made 100 turns. (2) The number of turns it will make before it stops if the coeff. of friction between the journals and their bearings = 0·07.

Answer.—(1) To find the reduction in speed after the wheel has made 100 turns, we must equate the work done against friction in 100 turns to the change of kinetic energy of the wheel during that time.

Let W = Weight of wheel = 4 tons = 4 × 2,240 lbs.
,, k = Radius of gyration of wheel = 5¼ ft.
,, n_1, n_2 = Initial and final revolutions per second.
,, d = Diameter of journals = ¾ ft.
,, μ = Coefficient of friction = 0·07.

Using equation (XI), we have :—

$$\textit{Change of } \mathrm{E_K} \textit{ of wheel} = \frac{2\,\pi^2 (n_1^2 - n_2^2)\, \mathrm{W}\, k^2}{g}.$$

And, from equation (II_b), Lecture VII., Vol. I.:—

Work lost in friction in one turn of journals $= \pi d \mu \text{R}$.

Where R is the resultant pressure on the bearings, and therefore = W in this case.

$$\therefore \left.\begin{array}{l}\textit{Work done against fric-}\\ \textit{tion in } 100 \textit{ turns}\end{array}\right\} = 100 \pi d \mu \text{W}.$$

$$\therefore \quad \frac{2 \pi^2 (n_1^2 - n_2^2) \text{W} k^2}{g} = 100 \pi d \mu \text{W}.$$

$$\therefore \quad n_1^2 - n_2^2 = \frac{100 d \mu g}{2 \pi k^2}.$$

Hence,

$$n_2^2 = n_1^2 - \frac{50 d \mu g}{\pi k^2}$$

$$\therefore \quad ,, = \left(\frac{80}{60}\right)^2 - \frac{50 \times {\cdot}75 \times {\cdot}07 \times 32}{\frac{22}{7} \times 5{\cdot}25 \times 5{\cdot}25}$$

$$,, = 1{\cdot}78 - {\cdot}97 = {\cdot}81.$$

Or,

$$n_2 = \sqrt{{\cdot}81} = {\cdot}9 \text{ rev. per sec., or } 54 \text{ revs. per min.}$$

$\therefore$ **Reduction in speed** $= n_1 - n_2 = 80 - 54 =$ **26 revs. per min.**

(2) Let n = number of turns made before stopping.

Then, in this case, the whole energy of the wheel when making 80 turns per minute is absorbed in friction at the journals.

$$\therefore \quad \frac{2 \pi^2 n_1^2 \text{W} k^2}{g} = n \pi d \mu \text{W}.$$

$$\therefore \quad n = \frac{2 \pi n_1^2 k^2}{\mu d g} = \frac{2 \times \frac{22}{7} \times \frac{80}{60} \times \frac{80}{60} \times 5{\cdot}25 \times 5{\cdot}25}{{\cdot}07 \times {\cdot}75 \times 32}$$

$$\therefore \quad n = 183\tfrac{1}{3} \text{ turns.}$$

Example VI.—A right cylinder of radius r, rolls, without slipping, down an inclined plane of height h. Find its velocity at the foot of the plane, and compare this with that which it would have had by merely sliding. Neglect frictional resistances in both cases.

Answer.—Let v = Velocity of $c.g.$ of cylinder at foot of plane.
„ ω = Angular velocity „ „
„ W = Weight of cylinder.
„ k = Radius of gyration about its own axis $= \frac{r}{\sqrt{2}}$.

Then, $\left.\begin{matrix}\textit{Total kinetic energy}\\ \textit{at foot of plane}\end{matrix}\right\} = \left\{\begin{matrix}\textit{Energy of Translation}\\ +\textit{Energy of Rotation.}\end{matrix}\right.$

But, $\left.\begin{matrix}\textit{Total energy at foot}\\ \textit{of plane}\end{matrix}\right\} = W\,h.$

Also, $\textit{Energy of Translation} = \frac{W\,v^2}{2\,g}.$

And, $\textit{Energy of Rotation} = \frac{W\,\omega^2\,k^2}{2\,g}.$

$$\therefore \quad W\,h = \frac{W\,v^2}{2g} + \frac{W\,\omega^2\,k^2}{2\,g}. \quad \therefore \quad 2\,g\,h = v^2 + \omega^2\,k^2.$$

$$\text{But,} \quad \omega = \frac{v}{r}, \text{ and } k = \frac{r}{\sqrt{2}}. \quad \therefore \quad \omega^2\,k^2 = \frac{v^2}{2}.$$

$$\therefore \quad 2\,g\,h = v^2 + \frac{v^2}{2}. \quad \therefore \quad v = \sqrt{\frac{4\,g\,h}{3}}.$$

Had the cylinder been allowed to *slide* down the plane *without rolling*, the velocity at foot of plane would have been :—

$$v = \sqrt{2\,g\,h}.$$

$$\therefore \left.\begin{matrix}\textbf{Vel. with rolling : Vel.}\\ \textbf{without rolling}\end{matrix}\right\} = \sqrt{\frac{4\,g\,h}{3}} : \sqrt{2\,g\,h} = \sqrt{2} : \sqrt{3}.$$

Of course, the kinetic energy of the body in both cases is the same, but in the second case the whole energy is translational, hence the reason for the greater speed in this case.

Example VII.—A weight, Q, draws up another weight, W, by means of an ordinary wheel and axle. The force ratio (Q : W) is 1 to 6, and the velocity ratio (vel. of Q : vel. of W) is 8 to 1. The diameter of the axle is 6 inches, and the radius of gyration of the wheel and its axle may be taken at 10 inches. Neglecting frictional resistances and the inertia of the ropes, determine the revolutions per minute of the machine after 10 turns have been made from a state of rest. Take the weight of the wheel and its axle = 2 W.

Answer.—We shall first answer this question in a general way.

Let W_1 = Weight of wheel and axle.
„ V = Velocity of effort, Q, in ft. per sec., after N turns.
„ v = Velocity of weight, W, „ „
„ R = Radius of wheel in feet.
„ r = Radius of axle „
„ k = Radius of gyration of wheel and axle in feet.
„ n = Revolutions per sec. of machine, after N turns.

Then, by the *Principle of Energy*, we get:—

Energy exerted = Work done + Change of kinetic energy.

$$\text{But, } \left.\begin{matrix}\textit{Energy}\\ \textit{exerted}\end{matrix}\right\} = Q \times \textit{Distance fallen in N turns of machine.}$$

$$,, \quad = Q \times 2\pi R N.$$

$$\textit{Work done} = \left\{\begin{matrix}W \times \textit{Distance raised in N turns of}\\ \textit{machine} = W \times 2\pi r N.\end{matrix}\right.$$

$$\left.\begin{matrix}\textit{Change of kinetic}\\ \textit{energy}\end{matrix}\right\} = \left\{\begin{matrix}\textit{Translational energy of Q and W + Rota-}\\ \textit{tional energy of wheel and axle.}\end{matrix}\right.$$

$$,, \quad ,, \quad = \frac{4\pi^2n^2Q \times R^2}{2g} + \frac{4\pi^2n^2W \times r^2}{2g} + \frac{4\pi^2n^2W_1 \times k^2}{2g}$$

$$,, \quad ,, \quad = \frac{2\pi^2n^2}{g}\left\{QR^2 + Wr^2 + W_1k^2\right\}$$

$$\text{Hence, } Q \times 2\pi RN = W \times 2\pi r N + \frac{2\pi^2n^2}{g}\left\{QR^2 + Wr^2 + W_1k^2\right\}$$

$$\text{Or, } (QR - Wr)N = \frac{\pi n^2}{g}\left\{QR^2 + Wr^2 + W_1k^2\right\}$$

This is the general expression from which n can be found when the other quantities are given.

From the question, we get:—$W = 6\,Q$; $W_1 = 2\,W = 12\,Q$; $N = 10$; $V = 8\,v$; $r = 3$ inches $= \frac{1}{4}$ foot; $R = \frac{V}{v}r = 8 \times \frac{1}{4}$ $= 2$ feet; $k = \frac{10}{12} = \frac{5}{6}$ foot.

$$\text{Hence, } \left.\begin{array}{r}\textit{Energy}\\ \textit{exerted}\end{array}\right\} = Q \times 2\pi RN = Q \times 2\pi \times 2 \times 10 = 40\pi Q \; \textit{ft.-lbs.}$$

$$\textit{Work done} = W \times 2\pi rN = 6Q \times 2\pi \times \tfrac{1}{4} \times 10 = 30\pi Q \quad ,,$$

$$\left.\begin{array}{l}\textit{Change of kinetic}\\ \textit{energy}\end{array}\right\} = \frac{2\,\pi^2 n^2}{g}\left\{ Q\,R^2 + W\,r^2 + W_1\,k^2 \right\}$$

$$,, \qquad ,, \qquad = \frac{2\pi^2 n^2}{32}\left\{ Q \times 2^2 + 6Q \times (\tfrac{1}{4})^2 + 12Q \times (\tfrac{5}{6})^2 \right\} ,,$$

$$,, \qquad ,, \qquad = \frac{\pi^2 n^2}{16} \times \frac{305}{24} Q \; \textit{ft.-lbs.}$$

$$\therefore \qquad 40\,\pi\,Q = 30\,\pi\,Q + \frac{305\,\pi^2 n^2}{16 \times 24} Q.$$

$$\therefore \qquad n^2 = \frac{10 \times 16 \times 24}{305 \times \frac{22}{7}} = 4.$$

$$\therefore \qquad n = 2 \textbf{ revolutions per second, } \text{or } 120 \text{ per min.}$$

Determination of the Energy of Flywheels.—Before the energy of a rotating body can be calculated at any given speed, it is necessary to know the radius of gyration of that body about the given axis of rotation. We have already shown how this quantity can be calculated in certain bodies which are of regular geometrical form; but many cases occur in the rotating parts of machines where the above methods of calculation would be most difficult, if not altogether impossible. Such is the case with most flywheels. The flywheel is a most important part of an engine, since it is a regulator of the speed. Owing to the great mass of its rim it naturally possesses great inertia, and is, therefore, capable of storing up a considerable amount of the energy developed in the cylinder, and of again imparting this stored energy to the moving parts during those portions of a revolution when the work done in the cylinder is less than the work being done outside. It is important to know the radius of gyration of the wheel, so that calculations relating to the storage and output of its energy can be effected. This radius of the wheel may be determined either approximately by calculation, or accurately by experimenting on the wheel itself, or with another similarly shaped wheel. We shall deal with these cases in turn.

(1) *By Approximate Calculation.*—Most flywheels consist of a heavy rim with comparatively light arms and nave; hence,

in calculations relating to the radii of gyration of such wheels, we may neglect the effects of the arms and nave, and consider only that of the heavy rim. Usually the rim is of a rectangular cross section.

Let R, r = Outside and inside radii of rim.

Then, from Table I., case V., of this Lecture, we get:—

$$k^2 = \tfrac{1}{2}(R^2 + r^2).$$

Substituting this in the equation for the kinetic energy of the wheel, we may obtain an approximate result.

Many engineers, however, further simplify their formula by taking for the radius of gyration the *mean* radius of the rim, and consider this quite near enough for most purposes. Thus:—

$$k = \tfrac{1}{2}(R + r).$$

The difference in the kinetic energy, as calculated from those two assumptions, may be shown as follows:—

Let W = Total weight of wheel.
„ ω = Angular velocity of wheel.

Then, according to the *first* assumption:—

$$\textit{The kinetic energy} = \frac{W\omega^2 k^2}{2g} = \frac{W\omega^2}{2g} \times \frac{R^2 + r^2}{2}.$$

And, according to the *second* assumption:—

$$\text{The kinetic energy} = \frac{W\omega^2}{2g} \times \frac{(R + r)^2}{4}.$$

$$\text{Hence, the difference} = \frac{W\omega^2}{2g}\left\{\frac{R^2 + r^2}{2} - \frac{(R + r)^2}{4}\right\}$$

$$\text{,,} \qquad \text{,,} \qquad = \frac{W\omega^2}{2g} \times \frac{(R - r)^2}{4}.$$

That is, the kinetic energy in the *first* case is greater than that in the *second* case by $\frac{W\omega^2}{2g} \times \frac{(R - r)^2}{4}$. This difference, however, becomes less as $R - r$ diminishes—that is, as r approaches R. On the other hand, it gets greater the thicker the rim. The radius of gyration in the first case—viz., $k^2 = \frac{1}{2}(R^2 + r^2)$, is too great; because the effect of the arms and nave is to reduce that radius, whereas the other result, $k = \frac{1}{2}(R + r)$, may be too small. It sometimes happens that a closer approximation may be obtained by taking the arithmetical *mean* of the above results, thus:—

$$\left.\begin{array}{c}\text{The kinetic energy}\\ \text{of the wheel}\end{array}\right\} = \frac{W\omega^2}{2g} \times \frac{1}{2}\left\{\frac{R^2 + r^2}{2} + \frac{(R+r)^2}{4}\right\}$$

$$\text{,,} \quad \text{,,} \quad = \frac{W\omega^2}{2g} \times \frac{\{3(R + r)^2 - 4Rr\}}{8}. \quad \text{(XIV)}$$

(2) *By Experiment on the Wheel.*—When accurate results are required, we may determine the radius of gyration of the wheel experimentally as follows:—

Disconnect the flywheel and its shaft from all other moving pieces, and see that the shaft runs smoothly in its bearings. Fit a flat pulley on the shaft and wind a few turns of flexible rope in a single layer round the same.* To the free end of this rope attach a weight sufficiently heavy to cause the flywheel to rotate at a uniform speed when started by the hand. This weight should just supply the energy absorbed by the friction of the shaft in its bearings and the bending of the rope. Now rewind the rope on the pulley and add another weight to its free end, so that the wheel will now start rotating when the weights are allowed to fall. Note the time taken by the weights in falling a known distance. The height through which the weights fall, and the diameter of the pulley being known, it is easy to calculate both the speed of the wheel and the falling weights, and hence their kinetic energies at the instant when the latter reach the ground.

Another method of allowing for the friction of the bearings, &c., is to use only one weight. Note the exact number of turns which the wheel makes (after the weight has ceased to act) until it comes to rest. Then neglecting the atmospheric resistance (which will be very small in an experiment of this kind) the work absorbed at the bearings will be equal to the kinetic energy of the wheel at the instant when the weight ceases to act.

These methods will be better understood when stated thus:—

---

* If the flywheel shaft be of sufficient diameter, this pulley may be dispensed with, and the rope need then be simply wound round the shaft. If a convenient direct drop for the weights cannot be arranged for, then the rope may pass round a guide pulley fixed to the roof, but in this case the kinetic energy of this pulley must be allowed for.

Let W = Weight of flywheel *in lbs.*
„ w = Weight producing motion of wheel.
„ w_1 = Weight required to balance friction.
„ k = Radius of gyration of flywheel *in feet.*
„ h = Height through which w and w_1 fall *in feet.*
„ D = Diameter of pulley keyed to shaft *in feet.*
„ d = Diameter of rope *in feet.*
„ t = Time taken by weight, w, in falling to the ground *in seconds.*
„ n = Number of revolutions *per second* which wheel is making at instant when w reaches the ground.
„ v = Velocity with which w and w_1 strike the ground.
„ N = Number of revolutions made by wheel after w ceases to act.

FIRSTLY.—*When w_1 is employed to balance the frictional resistances.* All the energy exerted by w is employed in giving kinetic energy to the wheel.

But, *Energy exerted* = $w\,h$.

$$\text{And,}\quad \left.\begin{matrix}\textit{Change}\\ \textit{of kinetic}\\ \textit{energy}\end{matrix}\right\} = \left\{\begin{matrix}\textit{Kinetic}\\ \textit{energy of}\\ \textit{wheel}\end{matrix}\right\} + \left\{\begin{matrix}\textit{Kinetic energy of } w \textit{ and}\\ w_1 \textit{ when they reach}\\ \textit{the ground}\end{matrix}\right.$$

$$,, \quad = \frac{W \times 4\,\pi^2 n^2 k^2}{2\,g} + \frac{(w + w_1)\,v^2}{2\,g}.$$

$$\therefore \quad w\,h = \frac{W \times 4\,\pi^2 n^2 k^2}{2\,g} + \frac{(w + w_1)\,v^2}{2\,g} \quad . \quad . \quad (1)$$

The revolutions, n, and the linear velocity, v, of the falling body at the instant when the latter reaches the ground can be determined as follows, when t, D, and d are known:—

$$\left.\begin{matrix}\textit{Number of revols. made by}\\ \textit{wheel during action of } w\end{matrix}\right\} = \frac{h}{\pi\,(D + d)} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

$$\therefore \quad \left.\begin{matrix}\textit{Average number of}\\ \textit{revols. per second}\end{matrix}\right\} = \frac{h}{\pi\,(D + d)\,t}$$

$$\therefore \quad n = \textit{Twice the average},$$

$$,, = \frac{2\,h}{\pi\,(D + d)\,t} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

$$\text{Similarly} \quad v = \textit{Twice average linear velocity of } w \textit{ and } w_1,$$

$$,, = \frac{2\,h}{t} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

$\therefore$ From equation (1) we get :—

$$w\,h = \frac{\mathrm{W} \times 4\pi^2 \times 4h^2 \times k^2}{2g \times \pi^2(\mathrm{D}+d)^2 t^2} + \frac{(w+w_1) \times 4h^2}{2g \times t^2}$$

$$\therefore \qquad \frac{8\,\mathrm{W}\,h\,k^2}{g\,(\mathrm{D}+d)^2\,t^2} = w - \frac{2\,(w+w_1)\,h}{g\,t^2}. \quad . \quad . \quad (\mathrm{XV})$$

SECONDLY.—*When the number of turns made by wheel after w ceases to act is known,*

Energy exerted $= w\,h.$

$$\left.\begin{array}{l}\textit{Work done on friction dur-}\\ \textit{ing last N revolutions of}\\ \textit{wheel}\end{array}\right\} = \left\{\begin{array}{l}\textit{Kinetic energy of wheel at}\\ \textit{instant when } w \textit{ ceases to}\\ \textit{act}\end{array}\right.$$

$$\text{,,} \qquad \text{,,} \qquad = \frac{\mathrm{W} \times 4\,\pi^2\,n^2\,k^2}{2\,g}.$$

But, by equation (2), the wheel makes $\dfrac{h}{\pi\,(\mathrm{D}+d)}$ revolutions during the action of w.

$$\therefore \left.\begin{array}{r}\textit{Work done on friction}\\ \textit{during action of } w\end{array}\right\} = \frac{\mathrm{W} \times 4\,\pi^2\,n^2\,k^2}{2\,g} \times \frac{\dfrac{h}{\pi\,(\mathrm{D}+d)}}{\mathrm{N}}.$$

From equation (3), we get :—

$$n = \frac{2\,h}{\pi\,(\mathrm{D}+d)\,t}.$$

$$\therefore \quad \textit{Work done} = \frac{\mathrm{W} \times 4\,\pi^2 \times \dfrac{4\,h^2}{\pi^2\,(\mathrm{D}+d)^2\,t^2} \times k^2}{2\,g} \times \frac{h}{\pi\,(\mathrm{D}+d)\,\mathrm{N}}$$

$$\text{,,} \qquad = \frac{8\,\mathrm{W}\,h^3\,k^2}{g\,\pi\,(\mathrm{D}+d)^3\,t^2\,\mathrm{N}}.$$

$$\left.\begin{array}{r}\textit{Change of}\\ \textit{kinetic energy}\end{array}\right\} = \frac{\mathrm{W} \times 4\,\pi^2\,n^2\,k^2}{2\,g} + \frac{w\,v^2}{2\,g}$$

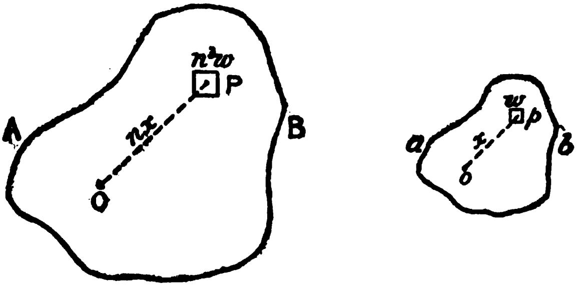
$$\text{,,} \qquad = \frac{8\,\mathrm{W}\,h^2\,k^2}{g\,(\mathrm{D}+d)^2\,t^2} + \frac{2\,w\,h^2}{g\,t^2}.$$

Hence, $$w\,h = \frac{8\,\mathrm{W}\,h^3\,k^2}{g\,\pi\,(\mathrm{D}+d)^3\,t^2\,\mathrm{N}} + \frac{8\,\mathrm{W}\,h^2\,k^2}{g\,(\mathrm{D}+d)^2\,t^2} + \frac{2\,w\,h^2}{g\,t^2}.$$

$$\therefore \frac{8\,\mathrm{W}\,h\,k^2}{g\,(\mathrm{D}+d)^2\,t^2}\left\{\frac{h}{\pi\,(\mathrm{D}+d)\,\mathrm{N}} + 1\right\} = w\left\{1 - \frac{2\,h}{g\,t^2}\right\}. \quad (\mathrm{XVI})$$

Equations (XV) and (XVI) enable us to find the radius of gyration, k, when the data are furnished by experiment.

If the wheel whose radius of gyration has to be found cannot be conveniently experimented on, then the radius of gyration of another similar wheel may be determined, and that for the first wheel calculated therefrom. It is easy to show generally that the moments of inertia of two similar bodies rotating about similarly placed axes are as the fifth powers of their like linear dimensions.



Moments of Inertia of Similar Bodies.

Let A B, and $a\,b$, be any two similar bodies whose axes O, and o, are similarly situated. Let the linear dimensions of the larger body be n times those of the smaller. Taking similar parts at P, and p, so that P is n times as large as p in each direction, it is evident that their masses will be in the proportion of $n^3 : 1$.

i.e., Mass of element at P : *Mass of corresponding element at* p $= n^3 m : m$. Also, if $o\,p = x$, then $\mathrm{O\,P} = n\,x$.

$\therefore$ *Mom. of inertia of* A B *about* O $= \Sigma\, n^3 m \times (n\,x)^2 = n^5\, \Sigma\, m\, x^2$,

and, *Mom. of inertia of* $a\,b$ *about* O $= \Sigma\, m\, x^2$.

$$\therefore \quad \left.\begin{array}{r}\text{Mom. of inertia of A B :}\\ \text{Mom. of inertia of } a\,b\end{array}\right\} = n^5 : 1. \quad . \quad . \quad . \quad . \qquad \text{(XVII)}$$

Thus, if two flywheels are made from the same drawing, but the scale in the one case be 4 inches to the foot, and in the other $1\frac{1}{2}$ inches to the foot, then their like linear dimensions will be inversely as the scales to which they are drawn, that is :—

Size of first wheel : Size of second wheel $= 1\frac{1}{2} : 4 = 3 : 8$.

$\therefore$ Mom. of inertia of first wheel : Mom. of inertia of second wheel $= 3^5 : 8^5 = 243 : 32768 = 1 : 134{\cdot}8$ nearly.

Centripetal and Centrifugal Force.—If a body is observed to be moving in a curvilinear path, either with uniform or variable speed, we at once infer that it is being continually acted upon by some deviating force directed towards the inside of the curve. In the case of a body moving in a circular path, that deviating force must be directed towards the centre of the circle. Hence, a body may be made to move in a circular path either by having it attached to a fixed point (the centre) by an inextensible string, or by compelling it to move in a circular groove. The necessary deviating force is supplied in the first case by the string attached to the body, while in the second case it is supplied by the sides of the groove. In either case this centrally-directed force is called the **Centripetal Force,** while its reaction is called the **Centrifugal Force.** These terms may be defined as follows :—

DEFINITION.—**Centripetal Force is that force which a guiding body exerts on a revolving body in order to compel the revolving body to move in its curvilinear path, and is always directed towards a fixed centre.**

DEFINITION.—**Centrifugal Force is the force with which a revolving body reacts on the body that constrains it to move in a curved path, and is equal and opposite in direction to the force with which the constraining body acts on the revolving body.**

i.e., **Centripetal Force = Centrifugal Force.**

We stated in Lecture XX. that when the velocity of a body changes, whether in magnitude or in direction, the velocity is said to be accelerated, and we have there shown how to measure this acceleration in the case of a particle moving with uniform speed in a circle. Thus, the radial or centripetal acceleration is there shown to be :—

$$a = \frac{v^2}{r}.$$

Where, v = Linear velocity of the particle in the circle,

and, r = Radius of the circle.

But an acceleration of a body can only be produced by the action of some force on it, and in the last Lecture we have shown how this force is measured when the weight of the body and the acceleration are known. Hence :—

$$F = \frac{w}{g} a.$$

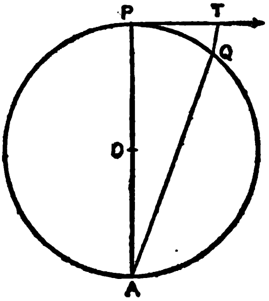
Let w = Weight of particle moving uniformly in a circle.
„ v = Linear velocity of particle in circle.
„ r = Radius of circle.
„ F = Centripetal or centrifugal force.
„ a = Centripetal acceleration $= \frac{v^2}{r}$.

Then, $$F = \frac{w}{g} a = \frac{w v^2}{g r} \quad . \; . \; . \; . \quad \text{(XVIII)}$$

We may, however, establish the same result in a different manner as follows :—

Let P be the position of the particle at any instant, and Q its position after a small interval of time, t. If no force acted on the body during that small interval of time, it would move along the tangent P T, and at the end of the interval be found at T, such that:—

$$PT = v t.$$



CENTRIPETAL FORCE.

But Q is its actual position; therefore T Q represents the deviation due to the centripetal force during that interval of time. Join Q A.

Then, $TQ = \frac{1}{2} a t^2$.

But, since P T and Q T are very small, T Q A will be very nearly a straight line.

$$\therefore \quad PT^2 = TA \times TQ \qquad \text{[Euc. III., 35]}$$
$$\text{„} \quad = (QA + TQ) \times TQ$$
$$\text{„} \quad = QA \times TQ + TQ^2.$$

In the limit, when t is infinitely small and, therefore, Q infinitely near to P, we may neglect TQ^2, and put $QA = PA = 2r$.

$$\therefore \quad v^2 t^2 = 2r \times \tfrac{1}{2} a t^2,$$

$$\therefore \quad a = \frac{v^2}{r}.$$

This is the same result as obtained by means of the Hodograph in Lecture XX.

$$\therefore \quad \textbf{Centrifugal force} = \frac{w}{g} a = \frac{w v^2}{g r}.$$

Let ω = Angular velocity of radius O P.

Then, $v = \omega r,$

$$\therefore \quad F = \frac{w \omega^2 r^2}{g r} = \frac{w \omega^2 r}{g}. \quad . \quad . \quad . \quad . \quad \text{(XIX)}$$

This shows that the centrifugal force is proportional to the square of the angular velocity of the particle, and to its distance from the centre of rotation.

We may now show that a similar expression holds good for the case of an extended rigid body turning about an axis.

Taking any particle of the body of weight w, and at a distance x from the axis of rotation, we get:—

$$\textit{Cent. force of the element} = \frac{w \omega^2 x}{g}$$

$$\therefore \quad \textit{Cent. force of whole body} = \frac{\omega^2}{g} \Sigma w x.$$

But, $\Sigma w x = W r.$

Where W = Weight of body,

And r = Distance of centre of gravity of body from axis of rotation.

$$\therefore \quad F = \frac{W \omega^2 r}{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XX)}$$

Hence, if the axis of rotation passes through the centre of gravity of the body, the centrifugal force is *nil*. If, however, the body be unsymmetrical about the axis of rotation, there may be, as explained in the next Lecture, a centrifugal *couple* tending to twist the axis of rotation and make the body rotate about some other axis.

EXAMPLE VIII.—A railway carriage weighing 4 tons is moving at the rate of 60 miles per hour round a curve ¼ mile in radius. Find the pressure on the rails due to centrifugal force; also, how much the outer rail should be higher than the inner rail in order that the pressure may be equally distributed on both? The distance between the rails is 4 feet 8½ inches.

ANSWER.—Here, $W = 4 \times 2240$ lbs.; $r = \frac{1}{4} \times 5280 = 1320$ feet; $v = \frac{60 \times 5280}{60 \times 60} = 88$ ft. per sec.

$$\therefore \left.\begin{array}{c}\textbf{Centrifugal}\\ \textbf{force}\end{array}\right\} = \frac{W v^2}{g r} = \frac{4 \times 2240 \times 88 \times 88}{32 \times 1320} = \textbf{1642·7 lbs.}$$

Hence, if both the inner and the outer rails were on a level, the flanges of the wheels would press on the latter with a force of 1642·7 lbs. By raising the outer line of rails above the level of the inner one, the carriage may be made to lie on an incline, and the outer rails thus relieved of the centrifugal pressure.

Let h = Height of the outer rail above level of the inner rail.
„ l = Distance between the rails = 4 ft. 8½ ins. = 56½ ins.
„ F = Centrifugal force on carriage = 1642·7 lbs.

Then, as a question on the *Inclined Plane*, we get :—

$$F : W = h : l.$$

$$\therefore \quad h = \frac{F}{W} \times l = \frac{1642 \cdot 7}{4 \times 2240} \times 56\tfrac{1}{2} = 10 \cdot 4 \text{ inches nearly.}$$

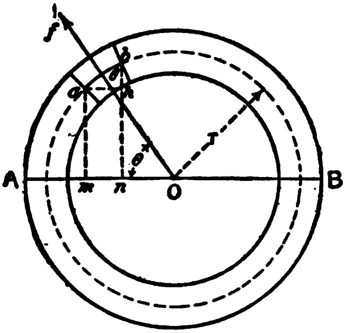
Straining Actions due to Centifrugal Forces.—Whenever a body rotates about an axis, the material of that body becomes strained by reason of the centrifugal forces set up. Thus, in the case of a flywheel or pulley, the centrifugal forces set up may be sufficient to tear the rim from the arms, the arms from the nave, or to burst the rim. In Lecture XVIII., Vol. I., we explained the effects of the centrifugal forces acting on a belt when moving over a pulley with a high velocity. We there showed that the tensions in the two parts of the belt were increased by the centrifugal action on that part of the belt which is in contact with the pulley. We shall now show that similar effects occur in a rapidly-revolving flywheel or pulley.

Suppose we have a flywheel built up of segments, each segment being attached to an arm, while they are also attached to each other by dowels and cotters, or bolts, &c. Let the weight of each segment be W; the distance of its centre of gravity from the axis of rotation, r, and the angular velocity of the wheel, ω. Then, neglecting the assistance afforded by the connection between the various segments, it is obvious that the tension in the arm to which the segment is attached is :—

$$P = \frac{W \omega^2 r}{g},$$

The arm must, therefore, be made strong enough to withstand this stress.

Again, in the case of a solid rim, the effect of the centrifugal forces is to burst it along a section made by a plane containing the axis of the shaft. Let the figure represent the rim of a flywheel. Then, in order to calculate the stress in its material at any section, A B, made by a plane containing the axis, O, consider the effects of a thin slice of the rim at $a\,b$.



Stress in Rim of Flywheel Due to Centrifugal Force.

Let W = Total weight of rim.
" r = Mean radius of rim.
" x = Length of small arc $a\,b$ of mean rim.
" ω = Angular velocity of wheel.

$$\text{Then, } \frac{\textit{Weight of slice } a\,b}{\textit{Weight of rim}} = \frac{\textit{Arc } a\,b \textit{ of mean rim}}{\textit{Circumference of mean rim}} = \frac{x}{2\,\pi\,r}.$$

$$\therefore \quad \textit{Weight of element } a\,b = \frac{W}{2\,\pi\,r} \times x.$$

The centrifugal force of the element at $a\,b$ is :—

$$\therefore \qquad f = \frac{W}{2\,\pi\,r} \times \frac{\omega^2\,r}{g} \times x = \frac{W\,\omega^2}{2\,\pi\,g}\,x.$$

This force acts through the *c.g.* of the element. Resolve f in directions parallel and perpendicular to A B. The latter component only is effective in producing stress at the sections A and B.

$$\therefore \left.\begin{array}{r}\textit{Stress at sections A and B due}\\ \textit{to cent. force on element } a\,b\end{array}\right\} = f \sin\theta = \frac{W\,\omega^2}{2\,\pi\,g} \times x \sin\theta.$$

$$\text{Where,} \qquad \theta = \angle A\,O\,f.$$

From a and b drop the perpendiculars $a\,m$, $b\,n$ on A B, and through a draw $a\,h$ perpendicular to $b\,n$. Then $\angle a\,b\,h = \theta$ and $x \sin\theta = a\,h = m\,n$.

$$\therefore \quad \left.\begin{array}{r}\textit{Stress at sections A and B due}\\ \textit{to cent. force on element } a\,b\end{array}\right\} = \frac{W\,\omega^2}{2\,\pi\,g} \times m\,n.$$

Continuing this reasoning for all the slices from A round to B, and adding the results, we get :—

$$\left.\begin{array}{l}\textbf{Total stress over}\\ \textbf{sections at A}\\ \textbf{and B}\end{array}\right\} = \frac{W\omega^2}{2\pi g}\Sigma m n = \frac{W\omega^2}{2\pi g} \times 2r = \frac{W\omega^2 r}{\pi g}.$$

Let A = Area of section of rim at A, or B, in *square inches.*
„ p = Stress in *lbs. per square inch* over section.
„ w = Weight of *a cubic foot* of material of rim.

$$\text{Then,} \quad p = \frac{\textit{Total stress over section at A or B}}{\textit{Area of section}}.$$

$$\text{Or,} \quad p = \frac{\frac{1}{2} \times \frac{W\omega^2 r}{\pi g}}{A} = \frac{W\omega^2 r}{2\pi g A}.$$

But, $W = \textit{Area of cross section of rim in sq. ft.} \times 2\pi r w.$

$$\therefore \quad W = \frac{A}{144} \times 2\pi r w.$$

Substituting this in the last equation, we get:—

$$p = \frac{w\,\omega^2 r^2}{144g} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XXI)}$$

Or, if n = Revolutions of wheel *per second,*
d = Diameter of rim *in feet,*
v = Velocity of rim *in feet per second* = ωr.

$$\left.\begin{array}{ll}\text{Then,} & p = \dfrac{w\,\pi^2 d^2 n^2}{144g} \text{ lbs. per square inch.}\\ \text{Or,} & p = \dfrac{w v^2}{144g} \quad \text{,,} \quad \text{,,} \quad \text{,,}\end{array}\right\} \text{(XXII)}$$

From this we see that the stress per square inch does not depend on the cross area of the rim nor the diameter of the wheel, but only on the density of the material and its speed. It will also be observed that the centrifugal force in the rim is similar in effect to a hydrostatic pressure on the inside of a cylindrical vessel.

EXAMPLE IX.—A flywheel, 21 feet in diameter, makes 100 revolutions per minute. The weight of a cubic foot of its material is 448 lbs. Find the intensity of stress on a transverse section of rim, assuming that it is unaffected by the arms. If the safe stress permissible in the material is 6,000 lbs. per square inch, what is the greatest speed at which the wheel can be run with safety?

Answer.—Here, $w = 448$ lbs. per cubic foot; $d = 21$ feet; $n = \frac{100}{60} = \frac{5}{3}$ revolutions per second.

Therefore, from equation (XXII), we get:—

$$\text{Stress in rim} = p = \frac{w\,\pi^2 d^2 n^2}{144\,g}$$

Or, $$p = \frac{448 \times \left(\frac{22}{7}\right)^2 \times 21^2 \times \left(\frac{5}{3}\right)^2}{144 \times 32} = 1176{\cdot}4 \text{ lbs. per sq. in.}$$

Next, let n = Maximum number of revolutions per second which the wheel can make without bursting.

Then, from the previous formula:—$p = \frac{w\,\pi^2 d^2 n^2}{144\,g}$,

We get, $$n^2 = \frac{144\,g\,p}{w\,\pi^2 d^2}, \qquad \text{or, } n = \frac{12}{\pi d}\sqrt{\frac{g\,p}{w}}.$$

Substituting $p = 6{,}000$, and the values for the other letters, we get:—

$$n = \frac{12}{\frac{22}{7} \times 21}\sqrt{\frac{32 \times 6000}{448}} = 3{\cdot}76 \text{ revs. per sec.} = 225{\cdot}6 \text{ per min.}$$

Note.—Students should refer to the author's *Text-Book on Steam and Steam Engines*, Lecture XVII., for a discussion of the effects of the inertia of the moving parts of an engine.

LECTURE XXII.—QUESTIONS.

1. Define the terms moment of inertia and radius of gyration of a body. Find the moment of inertia of rectangular lamina—first, with respect to one edge; secondly, with respect to a diagonal.

2. An axis is drawn through the centre of gravity of a body whose mass is M; a second axis is drawn parallel to the former and at a distance, h, from it. If I denotes the moment of inertia of the body with respect to the first axis, show that the moment of inertia with respect to the second axis is $\mathrm{I} + \mathrm{M}h^2$. A fine wire of uniform thickness is bent into the form of a circle whose radius is r; find its moment of inertia with respect to an axis passing at right angles to the plane of the circle through a point in the circumference. (S. & A. Theor. Mechs. Adv. Exam., 1878.) *Ans.* $\frac{3}{2}\mathrm{M}r^2$.

3. State and prove the theorem of moments of inertia for parallel axes. Find the moment of inertia of a cylinder about a line perpendicular to its axis through its mid point. (S. & A. Theor. Mechs. Hons. Exam.)

4. A wheel and axle are composed of the same specific gravity. The wheel is 4 feet radius, and 6 inches thick. The axle is 6 inches radius and 4 feet long. Find radius of gyration of the whole about the axis. *Ans.* $k = \sqrt{7{\cdot}125} = 2{\cdot}67$ ft.

5. The rim of a flywheel is rectangular in section, 6 inches wide, outside and inside radii 6 and 5 feet respectively. The nave is cylindrical, 2 feet long and 1 foot in diameter. There are eight cylindrical spokes of 4 inches diameter. Find the radius of gyration of the wheel. *Ans.* 3·75 ft.

6. Show that the kinetic energy of a body revolving with an angular velocity, ω, about a given axis is $\frac{1}{2}\mathrm{I}\omega^2$, where I denotes the moment of inertia of the body with reference to the axis. A flywheel has a mass of 30 tons, which may be supposed to be distributed along the circumference of a circle 8 feet in radius; it makes 20 revolutions a minute; find its kinetic energy in foot-pounds. (Adv. Theor. Mechs. Adv. Exam., 1883.) *Ans.* 295,000 ft.-lbs.

7. Find the moment of inertia of a rectangular lamina about an edge. A retangular lamina, whose shorter edges are 4 feet long, turns round one of its longer edges 50 times a minute. It weighs 441 lbs.; find its kinetic energy. (S. & A. Theor. Mechs. Adv. Exam., 1888.) *Ans.* 1008·3 ft.-lbs.

8. When a rigid body turns round an axis, what relation exists between its angular velocity and its kinetic energy? A rod of uniform density can turn freely round one end; it is let fall from a horizontal position; what is its angular velocity when it reaches its lowest position? Prove your equations. (S. & A. Theor. Mechs. Adv. Exam., 1878.) *Ans.* $\omega = \sqrt{\frac{3g}{l}}$.

9. How do you estimate the total energy possessed by a body when moving with both translation and rotation? Find the velocity of the centre (1) when a hoop, (2) when a disc, and (3) when a sphere rolls down an inclined plane of height, h. *Ans.* (1) $v = \sqrt{gh}$, (2) $v = 2\sqrt{\frac{gh}{3}}$, (3) $v = \sqrt{\frac{10gh}{7}}$.

10. Sketch, and explain the principle of the action of, a fly-press for stamping metals. If a velocity of 5 feet per second is given to the balls of such a press, and their motion is stopped after the screw has made one-quarter of a turn from the time that the die touches the metal, the pitch of the screw being $\frac{1}{4}$ inch; find the weight of the balls, so that the pressure exerted may be 4,000 lbs. *Ans.* 26·67 lbs. each.

11. Two weights of 100 lbs. each are placed at the ends of the arms of a fly-press, and are moving with a velocity of 12 feet per second. How many foot-pounds of work must be expended in bringing them to rest? Hence explain the mechanical action of the fly-press as a machine for punching or stamping metals. *Ans.* 450 ft.-lbs.

12. In a fly-press there are two weights, each of 60 lbs., placed at the ends of an arm which drives the screw; and the velocity of each weight at the instant of striking the blow is 10 feet per second. The die at the end of the screw moves through $\frac{1}{10}$ inch in coming to rest; what mean statical pressure does it exert on the metal subjected to the operation of stamping? *Ans.* 22,500 lbs.

13. In a fly-press for stamping metals a ball of 70 lbs. is placed at each end of the lever attached to the head of the screw. At the moment of striking the blow the weights have a velocity of 550 feet per minute, and the die at the end of the screw indents the metal to a depth of $\frac{1}{12}$ inch before coming to rest. What would be the mean statical pressure exerted on the metal? (S. & A. Exam., 1893.) *Ans.* 26,468·75 lbs.

14. Prove that the kinetic energy of a train of railway carriages moving with velocity, v, is $\left\{W + w\left(1 + \frac{k^2}{r^2}\right)\right\}\frac{v^2}{2g}$ ft. lbs., where w denotes the weight of the wheels and axles; W the weight of the rest of the train; r the radius of the wheels, and k the radius of gyration of a pair of wheels about their axis, the units being feet, lbs., and seconds. Determine the acceleration with which the train would freely descend an incline of inclination, α.

15. Describe and show by the necessary sketches the construction of a fly-press for punching holes in iron plates. In such a press the two balls weigh 30 lbs. each, and are placed at a radius of 30 inches from the axis of the screw, the screw itself being of 1 inch pitch. What diameter of hole could be punched by such a press in a wrought-iron plate of $\frac{1}{4}$ inch in thickness; the shearing strength of the metal being 22·5 tons per square inch? (Consider that the balls are revolving at the rate of 60 revolutions per minute when the punch comes into contact with the metal, and that the resistance of the plate is overcome in the first sixteenth of an inch of the thickness of the plate.) (S. & A. Adv. Exam., 1896.) *Ans.* 1·12 ins.

16. A pendulum bob weighing 20 lbs. is suspended by a wire, the length from the point of suspension to the centre of the bob being 16 feet. The pendulum swings through an angle of 30° on each side of the vertical; find its potential energy when in the highest position, and its velocity when passing the lowest point. (S. & A. Adv. Exam., 1895.) *Ans.* 42·88 ft. lbs.; 1372·16 ft. per second.

17. A flywheel weighs 10,000 lbs., and is of such a size that the matter composing it may be treated as if concentrated on the circumference of a circle 12 feet in radius; what is its kinetic energy when moving at the rate of 15 revolutions a minute? How many turns would it make before coming to rest if the steam were cut off and it moved against a friction of 400 lbs. exerted on the circumference of an axle 1 foot in diameter? (S. & A. Theor. Mechs. Adv. Exam., 1886.) *Ans.* 55,520 ft.-lbs.; 44·2 turns.

18. The sectional area of the rim of a cast-iron flywheel is 12 square inches, and the mean radius (or radius of gyration) is 25 inches; what is the kinetic energy at 150 revolutions per minute? What moment of constant magnitude, and acting through one-quarter revolution, would increase the speed to 155 revolutions per minute at the end of the quarter revolution? What would be the length of a solid wrought-iron shaft, 5 inches in diameter, rotating at the same speed and having the same kinetic energy? (C. & G. Mech. Eng. Hons. Exam., 1884.) *Ans.* 35,707,000 ft.-lbs.

19. Prove the formula for the energy stored up in a flywheel on the supposition that the whole of the material is collected in a heavy rim of given mean radius. Apply the formula to show (1) the effect of doubling the number of revolutions per minute; (2) the effect of doubling the weight; (3) the effect of increasing the mean radius in the proportion of 3 to 2. (S. & A. Exam., 1890.)

20. The rim of a flywheel weighs 9 tons, and the mean linear velocity of its mass is assumed to be 40 feet per second; how many foot-tons of work are stored up in it? If it be required to store the additional work of 9 foot-tons, what should be the increase of velocity? *Ans.* 225 ft.-tons; 0·79 ft. per second.

21. A flywheel weighs 2¼ tons, and its mean rim has a velocity of 40 feet per second. If the wheel gives out 10,000 foot-pounds of energy, how much is its velocity diminished? (S. & A. Exam., 1888.) *Ans.* 1·455 ft. per second.

22. A flywheel weighing 5 tons has a mean radius of gyration of 10 feet. The wheel is carried on a shaft of 12 inches diameter and is running at 65 revolutions per minute; how many revolutions will the wheel make before stopping if the coefficient of friction of the shaft in its bearing is 0·065? (Other resistances may be neglected.) (S. & A. Adv. Exam., 1896.) *Ans.* 354·66 turns.

23. A particle of given mass moves with a given velocity in a circle of given radius; state what is known as to the force which acts on the particle. Prove the statement. (S. & A. Adv. Theor. Mechs. Exam., 1896.)

24. If a locomotive weighing 55 tons runs round a curve of 1,200 feet radius at 20 miles per hour, what is its centrifugal force? How much higher in level should the outer rail be laid than the inner rail in order that the resolved part of the weight of the locomotive should balance this centrifugal force without pressure being exerted by the outer rail, the gauge being 4 feet 8½ inches? (C. & G. Mech. Eng. Hons. Exam., 1884.) *Ans.* 2760·6 lbs.; 1·27 inches.

25. Prove that a railway carriage running round a curve of radius, r, will upset if the velocity is greater than $\sqrt{\frac{g\,r\,a}{2\,h}}$, where a is the distance between the rails, and h the height of the centre of gravity of the carriage above the rails.

26. Show that by raising the outside rail of a railway track in going round a curve the tendency of the train to leave the rails is diminished, and that if θ be the inclination of the floor of the carriage to the horizontal, when there is no lateral pressure, $\tan\theta = \frac{v^2}{g\,r}$, where r is the radius of the curve, and v the velocity of the train. Hence show that on a 5-foot track, round a curve of one-eighth of a mile radius, that for a mean velocity of 30 miles an hour the outside rail ought to be raised 5½ inches above the level of the inside rail.

27. A body moves in a circle with a uniform velocity, show that it must be acted on by a constant force tending towards the centre, and find the magnitude of the force in terms of the radius of the circle, and of the mass and velocity of the body. A body weighing 2½ lbs. fastened to one end of a thread 4 feet long is swung round in a circle of which the thread is the radius; what will be its velocity when the tension of the thread is a force of 20 lbs.? ($g = 32$). *Ans.* 32 ft. per second.

28. A segment of a flywheel with the arm to which it is attached weighs 3,500 lbs., and the mass of the portion may be taken as collected at a distance of 8 feet from the axis of the wheel, which makes 40 revolutions per minute. What is the force tending to pull away the segment and arm from the boss of the wheel? You are required to write out a proof of the formula which you employ. (S. & A. Hons. Exam., 1889.) *Ans.* 15,365 lbs.

29. Show that the stress per square inch on the rim of a flywheel is equal to the momentum of the amount of rim (per square inch of section) which passes a fixed point in the unit of time. Find the limiting speed of periphery, the material being such that a bar of uniform section 900 feet long may be supported by tension. (S. & A. Mach. Const. Hons. Exam., 1885.) *Ans.* $30\sqrt{gr}$.

30. A flywheel 20 feet in diameter makes 80 revolutions per minute. Find the stress in its rim due to centrifugal forces, assuming that it is unaffected by the connection with the arms. The weight of a cubic foot of the material forming the rim is 500 lbs. What is the maximum speed at which the wheel can be safely run if the tensile strength of the material has not to exceed 6,000 lbs. per square inch? *Ans.* 762 lbs. per sq. in.; 224·5 revs. per min.

LECTURE XXIII.

CONTENTS. — Governing of Engines — Watt's Governor — Action of Watt's Governor—Theory of Watt's Governor—Conical Pendulum—Example I.—Common Pendulum Governor—Crossed-Arm Governor—Parabolic Governors—Galloway's Parabolic Governor—Porter's loaded Governor—Theory of Porter's Governor—Example II.—Spring loaded Governors—Proell's and Hartnell's Spring Governors—Macfarlane's Safety Governor—Willans' Spring Governor—Pickering Governor—Governing by Throttling and Variable Expansion—Shaft Governors—Relays—Knowles' Supplemental Governor—Inertia Governors—Flywheels—Balancing Machinery—Weston Self-balancing Centrifugal Machine—Questions.

Governing of Engines.*—For many purposes to which engines are applied, it is necessary that they should maintain a uniform speed. Owing to variations of load and of pressure on the piston, they must have some regulating device, in order to accomplish this object. Fluctuations of the speed of a steam engine are of two kinds. (1) Those which occur during the time of a revolution, and are *periodic*, being caused by the varying pressure on the piston, and obliquity of the connecting rod. (2) Those which are due to change of load, or boiler pressure, and are *not periodic*. To control the first of these as far as possible, an engine is fitted with a **Flywheel**, and for the second a **Governor** is also required.

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* The following is a list of books and papers treating of governors and governing:—

Paper on "The Electrical Regulation of the Speed of Steam Engines," by P. W. Willans. *Proc. Inst. C.E.*, 1885, vol. lxxxi., p. 166.

Paper on "A New Method of Investigation applied to the Action of Steam Engine Governors," by Prof. Dwelshauvers-Dery of Liége, translated by Michael Longridge. *Proc. Inst. C.E.*, 1888, vol. xciv., p. 210.

Paper on "The Cyclical Velocity-Variations of Steam and other Engines," by H. B. Ransom. *Proc. Inst. C.E.*, 1889, vol. xcviii., p. 357.

Paper on "The Application of Governors and Flywheels to Steam Engines," by Prof. Dwelshauvers-Dery, translated by Bryan Donkin. *Proc. Inst. C.E.*, 1891, vol. civ., p. 196.

Paper on "Flywheels and Governors," by H. B. Ransom. *Proc. Inst. C.E.*, 1892, vol. cix., p. 330.

Paper on "Steam Engine Governors and their Insufficient Regulating Action with Extreme Variations of Load," by Prof. Dwelshauvers-Dery, translated by Bryan Donkin. *Proc. Inst. C.E.*, 1892, vol. cx., p. 276.

Paper on "A Method of Testing Engine Governors," by H. B. Ransom. *Proc. Inst. C.E.*, 1893, vol. cxiii., p. 194.

A governor is a piece of mechanism which regulates the amount of steam supplied to the engine, to suit the work it is doing, whereas, as explained in the previous Lecture, a flywheel acts in virtue of its inertia, so as to distribute throughout a whole revolution the energy developed in the cylinder. The governor can have no effect whatever on the periodic variations of speed, since it can only act during the time that steam is being admitted to the cylinder. With regard to the irregular fluctuations of speed, due to a change of load, the flywheel makes them more gradual and thus gives the governor time to act. A great many varieties of governors have been invented since the introduction of the steam engine, such as hydraulic, centrifugal, inertia, and electrical governors. By far the greatest number, however, depend for their action on centrifugal force and inertia, and since these form useful examples of the practical application of the principles enunciated in the previous Lectures, we shall now confine our remarks to such governors.

Watt's Governor.—One of Watt's important inventions was his conical pendulum governor, as applied to his double-acting engine.[*] This governor consists of two arms, A A, carrying heavy balls, B B, and pivoted on a pin, P, passing through the centre of the vertical spindle, V S. The upper ends of these arms are bent, as shown on the figure, and are connected by short links, L L, to the sleeve, S. This sleeve is free to move vertically on the spindle, V S, but is made to rotate with it by a feather, F, and corresponding keyway. This sleeve acts on one end of the bell crank, B C, and thus moves the rod con-

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Paper on "The Mechanical and Electrical Regulation of Steam Engines," by John Richardson. *Proc. Inst. C.E.*, 1895, vol. cxx., p. 211.

Paper on "Governing of Steam Engines by Throttling and by Variable Expansion," by Capt. H. R. Sankey. *Proc. Inst. M.E.*, 1895, p. 154.

Paper on "Steam-Engine Governors," read before the Manchester Association of Engineers, by C. F. Budenberg, M.Sc. See *The Practical Engineer*, 17th April, 1891, vol. v., p. 258.

A series of articles on "Engine Governors," by R. G. Blaine, M.E., in *The Practical Engineer*, beginning 13th June, 1890, vol. iv., p. 386, and ending 24th April, 1891, vol. v., p. 277.

Article on "A New Shaft Governor," by E. J. Armstrong, in *The Practical Engineer*, 26th July, 1895, vol. xii., p. 71.

Article on "Shaft Governors," by E. T. Adams, in the *Electrical World* of New York. July, 1896.

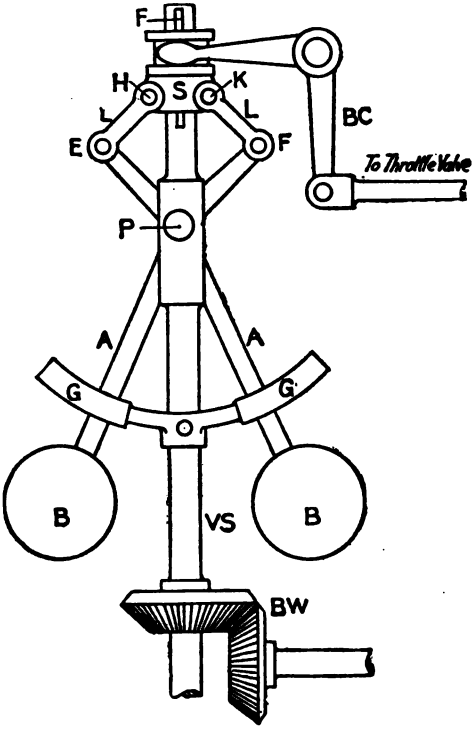
See Index for Governors in *Gas, Oil, and Air Engines*, by Bryan Donkin, published by Charles Griffin & Co.

The Steam Engine, by D. K. Clark (Blackie & Son), chap. v., on Governors, p. 65, half-vol. iii.

[*] See the Author's *Text-Book on Steam and Steam Engines*, Lecture II., for a description of Watt's engines. Also Lecture XIX., Volume I., of this book for an illustration of same.

nected to the throttle valve of the engine. The vertical spindle may be driven by the engine by means of a belt or rope passing round a pulley keyed on it, or by bevel wheels, as shown at B W. In order to relieve the pin, P, the arms are driven by the guides, G G, which are fixed to the vertical spindle.

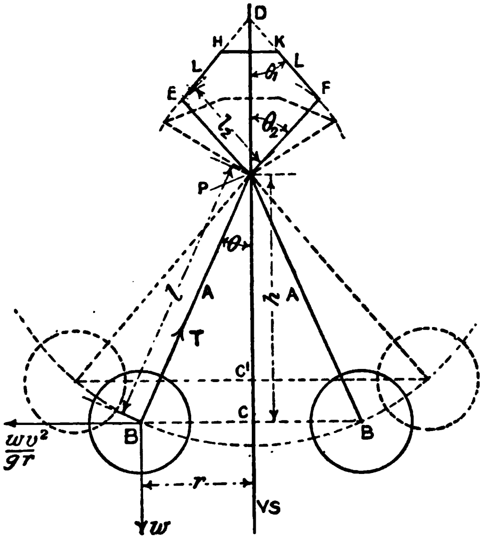
Action of Watt's Governor.—The governor is so adjusted, that when the engine is working at its normal speed, the balls rotate at a certain distance from the vertical spindle, and thus the throttle valve is kept sufficiently open to maintain that speed. Should the load be *decreased*, the speed of the engine, and therefore that of the governor balls, naturally becomes greater. This causes an increase of the centrifugal force of the balls, and therefore they diverge further, thereby pulling down the sleeve,



Watt's Pendulum Governor.

and partially closing the throttle valve, which diminishes the supply of steam and the power developed by the engine. On the other hand, should the load be *increased* the reverse action takes place, the balls come closer together, the sleeve is raised, the throttle valve opened wider, and more steam admitted to the engine. It will thus be seen that a change of speed must take place before the governor begins to act; further, that for any permanent change in the work to be done, there is a permanent alteration of speed. For each particular load on the engine, the throttle valve will be opened by a definite amount, which will be different for different loads, and each position of the valve has a corresponding position of the governor balls. But, as will be shown further on, each position of the balls corresponds to a definite speed, so that there will be a particular speed for each different load.

Theory of Watt's Governor—Conical Pendulum.—Let the balls



THEORY OF WATT'S GOVERNOR.

be rotating about the vertical spindle with a uniform velocity. Then the several forces acting on the different parts of the instrument are in equilibrium with each other. The arms, A, will describe the surface of a cone, B P B, whose height is P C, and for a given velocity of the balls there will be a definite height of this cone. It will be sufficient to consider one ball and arm, since what is true for one will be true for the other.

Let w = Weight of one ball in lbs.
,, v = Velocity of balls in *feet per second.*
,, h = Height, P C, of cone in *feet.*
,, l = Slant height, P B, of cone in *feet.*
,, r = Radius, B C, of base of cone in *feet.*
,, T = Tension in one arm, A.

There are three forces acting on the ball, B, viz.:—

(1) The weight, w, of the ball acting vertically downward.

(2) The centrifugal force, $w v^2 \div g r$, acting in its plane of rotation, and in the direction C B.

(3) The tension in the arm, A, acting in the direction B P.

These three forces keep the ball in equilibrium, and can, therefore, be represented, in magnitude and direction, by the three sides of a triangle taken in order. If we draw a triangle, having its sides parallel or perpendicular to the directions of these forces, the lengths of the sides of this triangle will be proportional to the forces respectively. Now, such a triangle exists in the figure itself—viz., the triangle P C B—the sides of which are parallel to the three forces:—

Hence, $$h : r = w : \frac{w v^2}{g r} = 1 : \frac{v^2}{g r}.$$

$$\therefore \qquad h = \frac{g r^2}{v^2}, \text{ and } \frac{r}{v} = \sqrt{\frac{h}{g}}.$$

If t = time in *seconds* of one complete revolution of balls,
n = number of revolutions per *second,*

Then, $$t v = 2 \pi r, \text{ and } n = \frac{1}{t}.$$

Substituting these in the previous equation we get the following important formulæ:—

$$= 2 \pi \frac{r}{v} = 2 \pi \sqrt{\frac{h}{g}}. \quad . \quad . \quad . \quad . \quad . \qquad \text{(I)}$$

That is, ***the period of rotation is proportional to the square root of the height of the cone.***

Also, $$n = \frac{1}{t} = \frac{1}{2 \pi} \sqrt{\frac{g}{h}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(II)}$$

Or, if N be the number of revolutions per *minute* = 60 n,

Then, $$N = \frac{30}{\pi}\sqrt{\frac{g}{h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{II}_a)$$

That is, *the number of revolutions, or the speed of the engine or governor, varies inversely as the square root of the height of the cone.*

Equation (II_a) may be written in this useful form:—

$$h = \frac{30^2 g}{\pi^2} \times \frac{1}{N^2} = \frac{2936}{N^2} \text{ feet.} \quad . \quad . \quad (\mathrm{II}_b)$$

Or, *the height of the cone depends only on the speed of rotation, and varies inversely as the square of the number of revolutions.*

Let the speed of the governor be altered from N_1 to N_2 revolutions per minute, then the heights of the cone corresponding to these speeds are:—

$$h_1 = \frac{2936}{N_1^2}, \text{ and } h_2 = \frac{2936}{N_2^2}.$$

Therefore, for a change of speed from N_1 to N_2 revolutions per minute the height of the cone will be altered by the amount:—

$$h_1 \sim h_2 = 2936\left(\frac{1}{N_1^2} \sim \frac{1}{N_2^2}\right) = \frac{2936\,(N_2^2 \sim N_1^2)}{N_1^2\,N_2^2}. \qquad (\mathrm{III})$$

If, however, the height of the governor be kept constant, and equal to $h = \frac{900\,g}{\pi^2 N_1^2}$, the centrifugal force will change from $\frac{w\,v_1^2}{g\,r}$ to $\frac{w\,v_2^2}{g\,r}$, or from $\frac{w\,r\,\pi^2 N_1^2}{900\,g}$ to $\frac{w\,r\,\pi^2 N_2^2}{900\,g}$, and the difference will produce a tension, or a thrust, in the links L L. If T_2 be the tension, or thrust, in one link L; l, l_1, l_2 the lengths of B P, E D, P E; and θ, θ_1, θ_2 their inclinations to the vertical, then by taking moments about P, we have:—

$$T_2 \times l_2 \cos(\theta_1 + \theta_2 - 90) = \frac{w\,r\,\pi^2}{900\,g}(N_2^2 \sim N_1^2) \times h,$$

Or, $$T_2 = \frac{w\,r\,\pi^2\,(N_2^2 \sim N_1^2)\,h}{900\,l_2\,g \sin(\theta_1 + \theta_2)}.$$

Now, the vertical force acting on the sleeve, which is available for overcoming friction, and may be called the *working effort* for that change of speed, is the vertical components of the stresses in the two links L L. These two stresses are equal.

$$\therefore \textit{ The working effort} = 2\,T_2 \cos \theta_1$$

$$\text{,,} \quad \text{,,} \quad = \frac{w\,r\,\pi^2\,h\,(N_2^2 \sim N_1^2)}{450\,l_2\,g} \times \frac{\cos \theta_1}{\sin(\theta_1 + \theta_2)}$$

$$\text{,,} \quad \text{,,} \quad = \frac{w\,r\,\pi^2 \dfrac{900\,g}{\pi^2 N_1^2}(N_2^2 \sim N_1^2) \cos \theta_1}{450\,l_2\,g \sin(\theta_1 + \theta_2)}$$

$$\text{,,} \quad \text{,,} \quad = \frac{2\,w\,r \cos \theta_1 (N_2^2 \sim N_1^2)}{l_2 \sin(\theta_1 + \theta_2)\,N_1^2} \quad \cdots\cdots \quad \text{(IV)}$$

It is usual for P E and E H to be made equal in length, and then $\theta_1 = \theta_2$ nearly, unless H K be great. In that case :—

$$\textit{The working effort} = \frac{2\,w\,r \cos \theta_2 (N_2^2 \sim N_1^2)}{l_2 \sin 2\,\theta_2\,N_1^2}$$

$$\left.\begin{aligned} \therefore \quad \text{,,} \quad \text{,,} \quad &= \frac{w\,r\,(N_2^2 \sim N_1^2)}{l_2 \sin \theta_2\,N_1^2} \\ \text{Or,} \quad \text{,,} \quad \text{,,} \quad &= \frac{w\,l \sin \theta\,(N_2^2 \sim N_1^2)}{l_2 \sin \theta_2\,N_1^2} \end{aligned}\right\} \quad \cdots\cdots \quad (\text{IV}_a)$$

If, further, $\theta = \theta_2$, which will always be the case when the sleeve is attached to the arm below the point of suspension, as in the next form of governor, then:—

$$\textit{The working effort} = \frac{w\,l\,(N_2^2 \sim N_1^2)}{l_2\,N_1^2} \quad \cdots\cdots\cdots \quad (\text{IV}_b)$$

It should be noted, however, that this is the effort exerted by the governor when it is just starting to move. The working effort becomes smaller and smaller as the balls rise, until, when the balls have attained the position corresponding to the new speed, it is *nil.*

The movement of the sleeve, corresponding to an alteration in the height of the cone, is best determined graphically by drawing the centre lines of the arms and links to scale for different positions of the balls.

EXAMPLE I.—Find the rise of the balls of a pendulum governor, when its speed is increased from 60 to 62 revolutions per minute. Find also the height of the cone of revolution at the lower speed, and the working effort, if the balls weigh

22 lbs. each, and each arm is jointed to the link at two-thirds of its length below its point of suspension.

Here, $N_1 = 60$, $N_2 = 62$, rise of balls $= h_1 - h_2$, $w = 22$, and $\frac{l}{l_2} = \frac{3}{2}$. Therefore, from equation (III) we get:—

$$h_1 - h_2 = 2936\left(\frac{N_2^2 - N_1^2}{N_1^2 \times N_2^2}\right) = 2936\left(\frac{62^2 - 60^2}{60^2 \times 62^2}\right).$$

$$\therefore \quad h_1 - h_2 = \cdot 0518 \text{ foot or } \cdot 62 \text{ inch.}$$

Also from equation (II_b):—

$$h_1 = \frac{2936}{N_1^2} = \frac{2936}{3600}.$$

$$\therefore \quad h_1 = \cdot 816 \text{ foot} = 9 \cdot 79 \text{ inches.}$$

And from equation (IV_b) :—

$$\textit{The working effort} = \frac{w\,l\,(N_2^2 \sim N_1^2)}{l_2 N_1^2}$$

$$\text{,,} \quad \text{,,} \quad = \frac{22 \times 3\,(62^2 - 60^2)}{2 \times 60^2}$$

$$\text{,,} \quad \text{,,} \quad = 2 \cdot 237 \text{ lbs.}$$

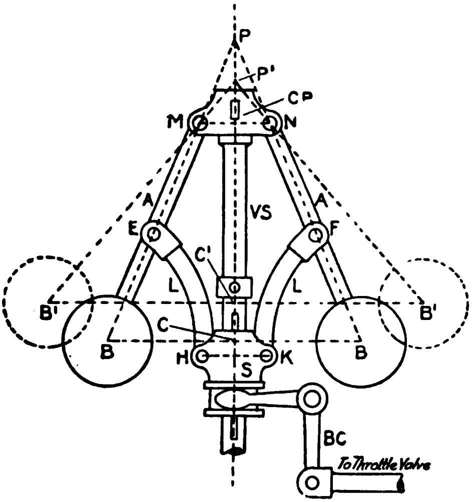
In this case, as we assume $l_1 = l_2$ and $\theta_1 = \theta_2$, the travel of the sleeve will be twice the rise of the point where the link joins the arm, and this will be two-thirds of the alteration in height.

$$\therefore \textit{Travel of sleeve} = 2 \times \frac{2}{3}(h_1 - h_2) = \frac{4}{3} \times \cdot 62 = \cdot 827 \text{ inch.}$$

Common Pendulum Governor.—A common modification of Watt's governor is shown by the following figure. Here, the arms A A, carrying the balls B B, instead of being jointed together by a pin passing through the vertical spindle V S, are pivoted at M and N to a cross-piece, C P, which is rigidly connected to the spindle. The links L L, carrying the sleeve S, are attached to the arms at the points E and F.

The formulæ deduced for Watt's pendulum governor are equally applicable to this case. The only thing requiring special attention here, is the height of the cone of revolution. The vertex of the cone is always at the point where the centre lines of the arms meet. In this case, the arms terminate at M and N, which are at a short distance from V S, and thus the vertex of the cone will be a variable point on the centre line of the vertical spindle. When the balls are in the position shown by the full lines, the vertex is at P, and the height of the cone is P C; but, when the balls move into the new position, shown by dotted lines, the vertex of the cone is at P′, and the height of the cone is P′ C′.

The effect of suspending the arms at a short distance from the vertical spindle, is to cause the movement of the sleeve to be less for a given variation in the height of the governor, than would be the case were the centres of suspension in the vertical spindle. It will be apparent from the figure, that the effective variation of height is $CC' = (PC - P'C') - PP' = (h_1 - h_2) - PP'$, instead of $h_1 - h_2$, as in the previous case. Hence, this governor is *less sensitive* than the former, since the speed must vary between greater limits for a given movement of the sleeve. The *sensitiveness* of a governor depends on the movement given to the sleeve for a given variation in speed, and also on the smallness of the time taken by the governor in adapting itself to its new position. In order, therefore, to increase the sensitiveness of this form of governor it is necessary that the points M and N should be as near the vertical spindle as possible.

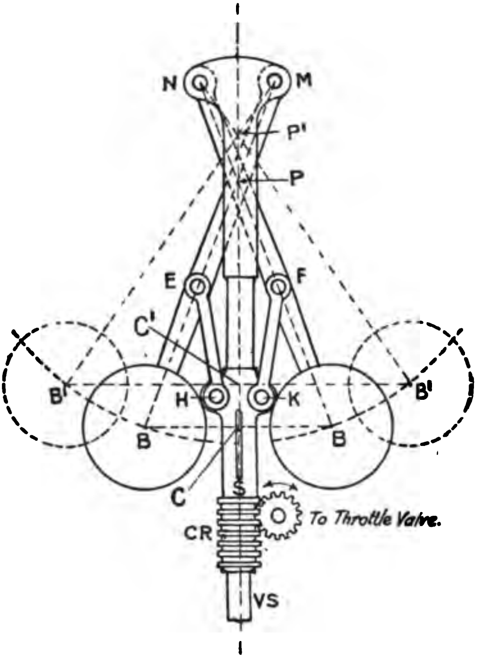


COMMON PENDULUM GOVERNOR.

Crossed-Arm Governor.—The special feature in which this governor differs from the former ones is, that the arms are suspended from pins placed on the opposite sides of the spindle to that of their balls. From an inspection of the figure, it will be apparent that crossing the arms in this manner causes a greater movement of the sleeve for a given variation in the height than in the pendulum governor. The rise of the balls in this case is :—

$$C\,C' = P\,C - P'\,C' + P\,P' = (h_1 - h_2) + P\,P'.$$

The sensitiveness of this governor is therefore much greater than either of the two previous forms. By properly proportioning the lengths of the arms and N M, so that the balls move out and in, along a curve which is approximately a parabola, this governor may be made almost isochronous, and, therefore, extremely sensitive.* It will be noticed that the sleeve of the



CROSSED-ARM GOVERNOR.

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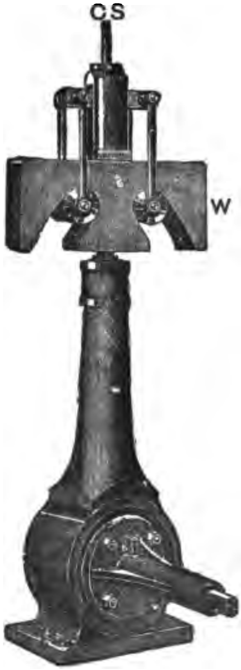
[*] A governor is said to be *isochronous* when its speed of rotation (and, therefore, the height of the cone) is the same for all positions of the balls within its range.

governor shown, has a circular rack O R, which gears with a pinion on the throttle valve spindle.

Parabolic Governors.*—Governors have been so constructed that their balls were guided to move in a truly parabolic path, and thus be absolutely isochronous, but owing to their complication they have not come into general use. With any centrifugal governor, the speed must increase somewhat before the extra centrifugal force is able to overcome the friction resisting the motion of the links, sleeve, valve, &c., and if it be absolutely isochronous, whenever the friction is overcome the balls would rise right up to the top of their range, and remain there until the speed has fallen sufficiently for gravity to reassert itself and overcome the friction, which would now tend to keep the balls up. They would then come down to the bottom of their range, and there would thus be continual hunting. Such a governor would therefore be wanting in stability or steadiness.



Loaded Parabolic Governor, by Galloways, Ld.

Galloway's Parabolic Governor.—From the illustration it will be seen, that in this type two cylindrical rollers take the place of the ordinary balls in the previously mentioned governors. These rollers are suspended at each end by links from a crosshead fixed to the top of the governor spindle, and naturally rise and fall in circular arcs with these links as radii. They move along parabolic slots cut in a weight W, which rotates with the spindle, but is free to rise and fall along the same. By this arrangement, the moment of the centrifugal force of the rollers is balanced by that of the weight at nearly the same speed for all positions. Hence, this governor may be considered practically insochronous. To the bottom of the slotted weight there is sometimes attached a sleeve terminating in a collar, which engages the forked lever connected to the throttle valve, or expansion gear; but in this case, a central spindle C S, which is carried up inside the main governor spindle, rises and falls with the weight W, and acts directly on an equilibrium valve. The governor is driven through gearing contained in the cast-iron box seen at the foot of the vertical column.

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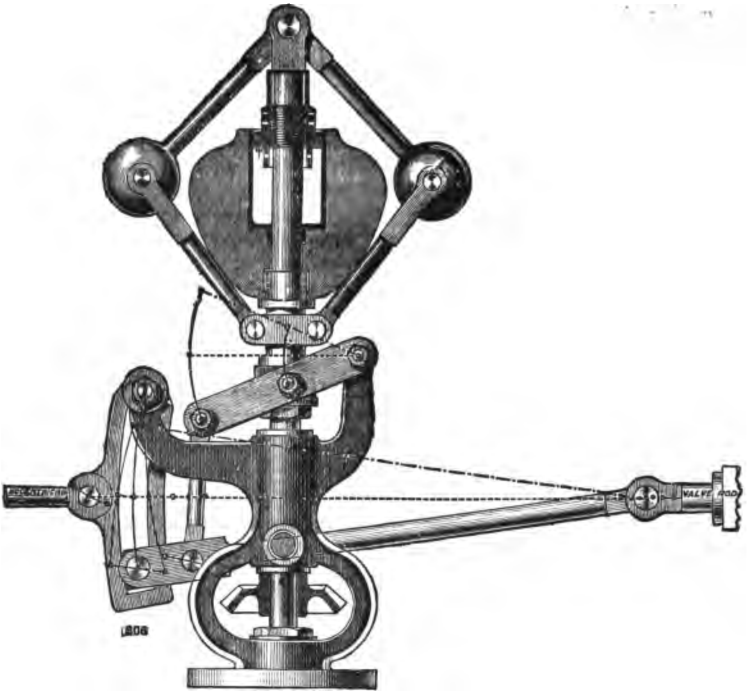
* See the Appendix to *The Steam Engine*, by Prof. Rankine (Chas. Griffin & Co.), and Chapter XV. of *Practical Treatise on the Steam Engine*, by Arthur Rigg (E. & F. Spon), for descriptions of guided parabolic governors.

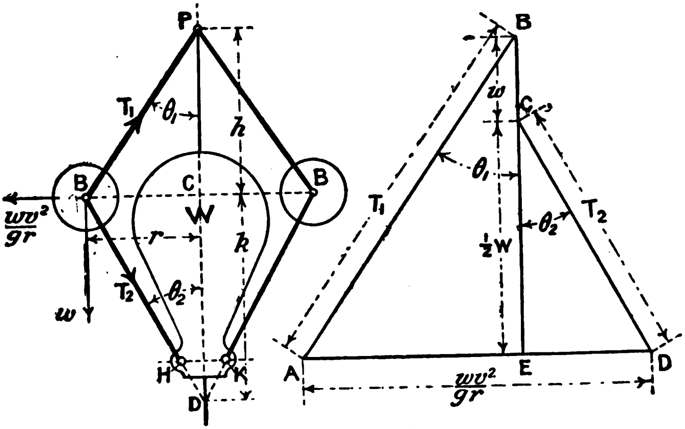
Porter's Loaded Governor.—From equation (IV) and Example I. we see that the simple pendulum governors possess a comparatively small working effort, unless the balls are very heavy. To overcome this objection Porter made the balls smaller, and loaded the sleeve with a heavy weight. This increases the height of the cone, corresponding to any particular speed, and all the forces concerned, and thus gives a greater working effort. It can be used both in connection with throttle valves and some forms of expansion gear. To minimise the oscillations of the ordinary Porter governor, Messrs. Clayton & Shuttleworth have made a cylindrical hole in the top of the central weight, and fixed a piston on the vertical spindle, thus forming a simple air cushion.

PORTER LOADED GOVERNOR, BY TANGYES, LIMITED.

Theory of the Porter Governor.—Each of the balls is in equilibrium under the action of four forces acting in a plane passing through the axis of rotation. These forces are:—(1) the weight of the ball w, (2) the centrifugal force $w v^2 \div g r$, and (3) the tensions in the two links, T_1 and T_2. Let A B C D A be a polygon representing these forces, A B being parallel and equal to T_1, B C to w, C D to T_2, and D A to the centrifugal force. If B C be produced to meet A D in E, then C E is equal to the vertical component of T_2, and must therefore be equal to half the load W, since this weight is supported by the vertical components of the tensions in the two bottom links.



PORTER GOVERNOR, BY CLAYTON & SHUTTLEWORTH.



ACTION OF THE PORTER GOVERNOR.

If the inclination of the links B P, B H to the vertical, be θ_1 and θ_2 respectively, and the heights C P, C D be denoted by h and k, then, the other letters being the same as before, we have:—

$$T_1 \sin \theta_1 + T_2 \sin \theta_2 = \frac{w v^2}{g r}.$$

$$T_1 \cos \theta_1 = w + \tfrac{1}{2} W, \text{ or } T_1 = \frac{w + \frac{1}{2} W}{\cos \theta_1}.$$

And, $$T_2 \cos \theta_2 = \tfrac{1}{2} W, \qquad \text{or } T_2 = \frac{\frac{1}{2} W}{\cos \theta_2}.$$

$$\therefore \frac{w + \frac{1}{2} W}{\cos \theta_1} \sin \theta_1 + \frac{\frac{1}{2} W}{\cos \theta_2} \sin \theta_2 = \frac{w v^2}{g r}.$$

But, $$\tan \theta_1 = \frac{r}{h}; \qquad \tan \theta_2 = \frac{r}{k}; \text{ and } v = 2 \pi r n.$$

$$\therefore \qquad (w + \tfrac{1}{2} W) \frac{r}{h} + \tfrac{1}{2} W \frac{r}{k} = \frac{w \times 4 \pi^2 r^2 n^2}{g r}.$$

Or, $$(w + \tfrac{1}{2} W) \frac{1}{h} = \frac{w \times 4 \pi^2 n^2}{g} - \frac{\frac{1}{2} W}{k}.$$

$$\therefore \qquad h = \frac{w + \frac{1}{2} W}{\dfrac{4 \pi^2 n^2 w}{g} - \dfrac{\frac{1}{2} W}{k}}.$$

Hence, $$h = \frac{(2 w + W) g k}{8 \pi^2 n^2 w k - W g}$$

Or, since $n = \frac{N}{60}$, $$h = \frac{450 (2 w + W) g k}{\pi^2 N^2 w k - 450 W g} \qquad \text{(V)}$$

This governor is usually constructed with all four links of equal length; then $k = h$, and $\theta_1 = \theta_2 = \theta$, very nearly, unless the distance H K is great, and in our further investigations we shall assume that this is so.

In this case we have:—

$$T_1 \sin \theta + T_2 \sin \theta = \frac{w v^2}{g r}.$$

$$T_1 \cos \theta = w + \tfrac{1}{2} W, \text{ or } T_1 = \frac{(w + \frac{1}{2} W)}{\cos \theta},$$

And, $$T_2 \cos\theta = \tfrac{1}{2}W, \qquad \text{or } T_2 = \frac{\frac{1}{2}W}{\cos\theta}$$

$$\therefore \qquad (w + \tfrac{1}{2}W)\tan\theta + \tfrac{1}{2}W\tan\theta = \frac{w v^2}{g r}.$$

But, $\tan\theta = \frac{r}{h}$ and $v = 2\pi r n$.

Hence, $$(w + W)\frac{r}{h} = \frac{w \times 4\pi^2 r^2 n^2}{g r}.$$

$$\left.\begin{aligned} \therefore \qquad h &= \frac{w + W}{w} \times \frac{g}{4\pi^2 n^2}, \\ \text{Or,} \qquad h &= \frac{w + W}{w} \times \frac{900 g}{\pi^2 N^2} = \frac{w + W}{w} \times \frac{2936}{N^2} \end{aligned}\right\} \text{(VI)}$$

We might also have arrived at this result by putting h for k in equation (V).

If the speed of rotation change from N_1 to N_2 revolutions per minute, the corresponding heights of the governor will be:—

$$h_1 = \frac{w + W}{w} \times \frac{900 g}{\pi^2 N_1^2}; \qquad \text{and } h_2 = \frac{w + W}{w} \times \frac{900 g}{\pi^2 N_2^2}.$$

The alteration in the height of the cone of revolution would therefore be:—

$$\left.\begin{aligned} h_1 \sim h_2 &= \frac{w + W}{w} \times \frac{900 g}{\pi^2} \times \left(\frac{N_2^2 \sim N_1^2}{N_1^2 N_2^2}\right) \\ \text{Or, } h_1 \sim h_2 &= 2936\,\frac{w + W}{w}\left(\frac{N_2^2 \sim N_1^2}{N_1^2 N_2^2}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(VII)}$$

With the arrangement of links usually adopted in this governor, the travel of the sleeve is twice the change in the height of the balls and equal to $2\,(h_1 \sim h_2)$.

Using the simpler equation (VI) we see that:—

$$W = \frac{\pi^2 N^2 w h}{900 g} - w.$$

And, therefore, if the speed alter from N_1 to N_2 revolutions per minute, the load necessary to keep the height of the cone constant, and equal to h, $\left(= \frac{w + W}{w} \times \frac{900 g}{\pi^2 N_1^2}\right)$, will change from—

$$W = \frac{\pi^2 N_1^2 w h}{900 g} - w \text{ to } W' = \frac{\pi^2 N_2^2 w h}{900 g} - w.$$

But if the actual load be W, the difference $W' \sim W$ is available for overcoming friction and moving the valve. This may be called the *working effort* for that change in speed, and is equal to:—

$$W' \sim W = \frac{\pi^2 w h}{900 g}(N_2^2 \sim N_1^2)$$

Or, $$W' \sim W = \frac{\pi^2 w \dfrac{w + W}{w} \times \dfrac{900 g}{\pi^2 N_1^2}(N_2^2 \sim N_1^2)}{900 g}$$

$$\therefore \quad W' \sim W = (w + W)\frac{N_2^2 \sim N_1^2}{N_1^2}. \quad . \; . \; . \; . \; . \quad \text{(VIII)}$$

EXAMPLE II.—The balls of a Porter governor weigh 4 lbs. each, and the central weight 36 lbs. If all the links are of equal length, find the height of the governor when revolving 240 times per minute. If the speed increase to 248 revolutions per minute what will be the working effort and the rise of the balls and the sleeve?

Here, $N_1 = 240$; $N_2 = 248$; $w = 4$; and $W = 36$.

From equation (VI), $$h_1 = \frac{w + W}{w} \times \frac{2936}{N_1^2},$$

Or, $$h_1 = \frac{4 + 36}{4} \times \frac{2936}{240 \times 240}.$$

$$\therefore \quad h_1 = \cdot 51 \text{ foot or } 6\cdot 12 \text{ inches.}$$

From equation (VIII), *The working effort,* $$= (w + W)\frac{N_2^2 - N_1^2}{N_1^2},$$

„ „ $$= (4 + 36)\frac{248^2 - 240^2}{240^2},$$

„ „ $$= 2\cdot 71 \text{ lbs.}$$

And from equation (VII), $$h_1 - h_2 = 2936\,\frac{w + W}{w}\left(\frac{N_2^2 - N_1^2}{N_1^2 N_2^2}\right),$$

Or, $$h_1 - h_2 = 2936 \times \frac{4 + 36}{4} \times \frac{248^2 - 240^2}{240^2 \times 248^2}.$$

$$\therefore \quad h_1 - h_2 = \cdot 0324 \text{ foot or } \cdot 389 \text{ inch.}$$

∴ ***Travel of sleeve*** $= 2\,(h_1 - h_2) = 2 \times \cdot 389,$

„ $= \cdot 778$ inch.

On comparing these results with those of Example I., it will be noticed that while both governors weigh about the same, the loaded governor has a working effort of about 20 per cent. greater than that of the Watt governor, but its travel and height are less. It will also be seen from equations (III) and (VII), by putting $N_2 = c N_1$, where c is a constant, that the change in height corresponding to a given percentage variation in speed gets smaller as the speed increases.

Spring Loaded Governors.—Soon after the introduction of the Porter governor, Mr. John Richardson, of Messrs. Robey & Co., Lincoln, designed one in 1869, in which a spring was substituted for the weight. This improvement produces a greater working effort with less weight, bulk, and cost. A spring has less inertia, and acts much more quickly than a weight, and it has also a certain amount of cushioning action. A governor loaded with a spring can act in any position, whereas one with a weight must work vertically. The equations for a spring governor may be obtained in the same way as for the weighted governor, but the load W will be different for different positions of the balls, owing to the varying compression of the spring.

Proell and Hartnell's Spring Governors.*—The first of the two following figures illustrates the well-known Proell spring governor. It will be observed that a helical spring, contained in a cylindrical case, surrounds the governor spindle, and bears upon the inner ends of the two bell crank levers, which are connected to the arms carrying the governor balls. The dotted lines show the positions of the balls for a speed above the normal. As they move out to this position the spring is compressed and the sleeve is raised. It will further be noticed that the links are so proportioned that the balls diverge in nearly a straight line. Consequently, when working vertically, the balls do not move either with or against gravity.

It will be observed that there are no less than three pin joints on each side of the Proell governor above the sleeve, at each of which there must be friction. In Hartnell's governor, illustrated by the next figure, there is but one.†

Here, the governor balls are fixed directly to the outer ends of the bell crank levers, the inner ends of which bear upon a collar on the upper end of the movable tube or sleeve, MS.

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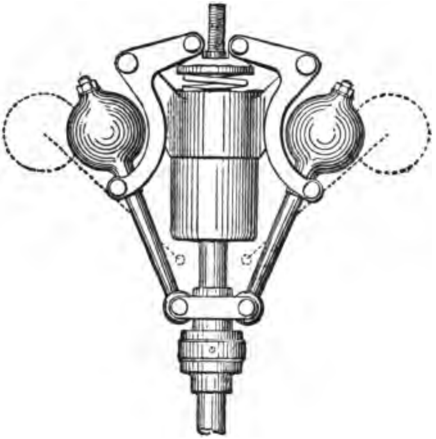
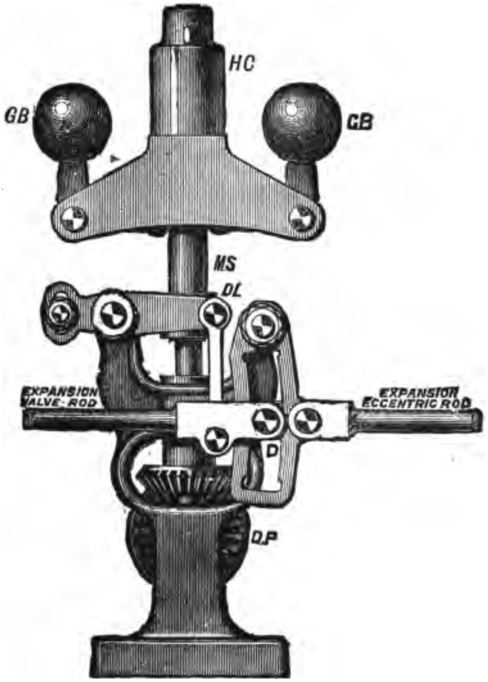
* The figure of Proell's governor is from *The Proc. Inst. C.E.*, vol. cxx., Session 1895-96, by kind permission of the Council, from a paper read by John Richardson, M.Inst.C.E., on "The Mechanical and Electrical Regulation of Steam Engines," which the student should consult.

† See Lecture XVIII. of the Author's *Text-Book on Steam and Steam Engines* for a description of an engine to which this governor is applied.



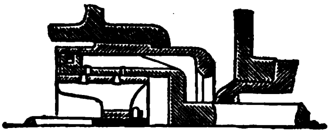
PROELL'S SPRING GOVERNOR.



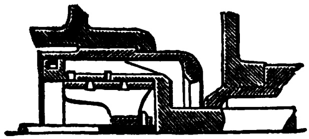
HARTNELL'S SPRING GOVERNOR, BY MARSHALL, SONS & CO., LTD.

Between the top of this collar and the upper end of the hollow casting H C, which is keyed to the top of the governor spindle, there is placed a strong helical spring. The lower end of M S has a double collar engaged by a forked lever, which works another lever connected to the drag link D L, actuating the expansion valve rod. In the discussion on Mr. Richardson's paper already referred to, Mr. Kuhne states that a Proell governor, weighing about 2 cwts., developed a working effort of 21 lbs. for an increase of 2 per cent. in the speed, and that a Porter governor to do the same would weigh $\frac{3}{4}$ ton. Also, that the force required to compress the spring when the balls were quite open was 1,781 lbs.

Macfarlane's Safety Governor.—This governor is fitted inside the steam passage, and acts directly on the grating throttle valve C. There are two weights D, working on centres at E, which move the sliding piece F, attached to the throttle valve spindle G. This spindle is forced to the right by the spring H, placed inside the governor spindle I, and is pulled to the left as the balls fly out. The governor spindle passes through the stuffing box T, and is driven by the pulley X. In addition to the throttle valve there is a stop valve K, and seat O; and the bearing L for the throttle valve forms part of this. From the left of the two small figures it will be seen that the holes in C and L are opposite to each other when the engine is working normally, but should the speed increase, C is pulled to the left and cuts off towards that side. Should the belt which drives the governor slip or break, or the governor stop from any cause, the throttle valve is forced to the right by the spring and shuts off steam completely, as shown on the other figure. The stop valve is shown closed in the large figure.



NORMAL POSITION OF THROTTLE VALVE.



CLOSED POSITION OF THROTTLE VALVE IF BELT BREAKS.

Willans' Spring Governor.*—In the previous cases, the pressure of the spring has to be transmitted through the pin joints of the

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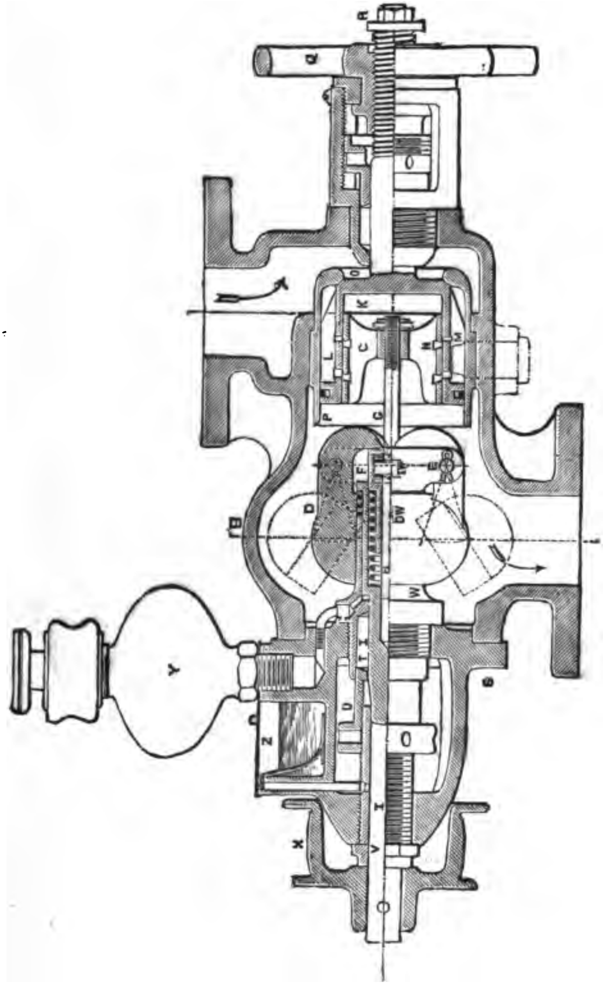
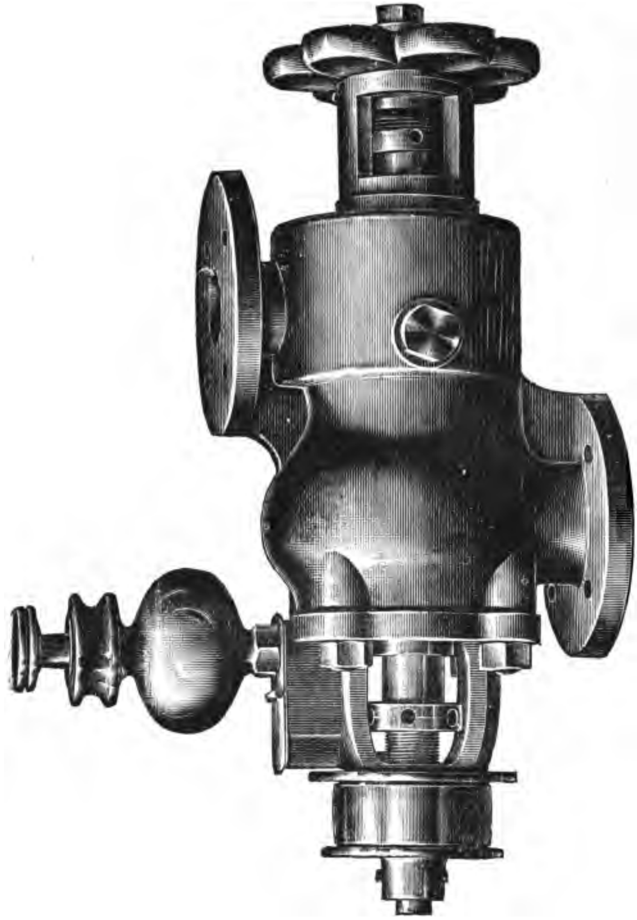
* For a description of Willans' central-valve triple expansion engine, to which this governor is fitted, see Appendix III. of the Eleventh Edition of the Author's *Text-Book on Steam and the Steam Engine.*

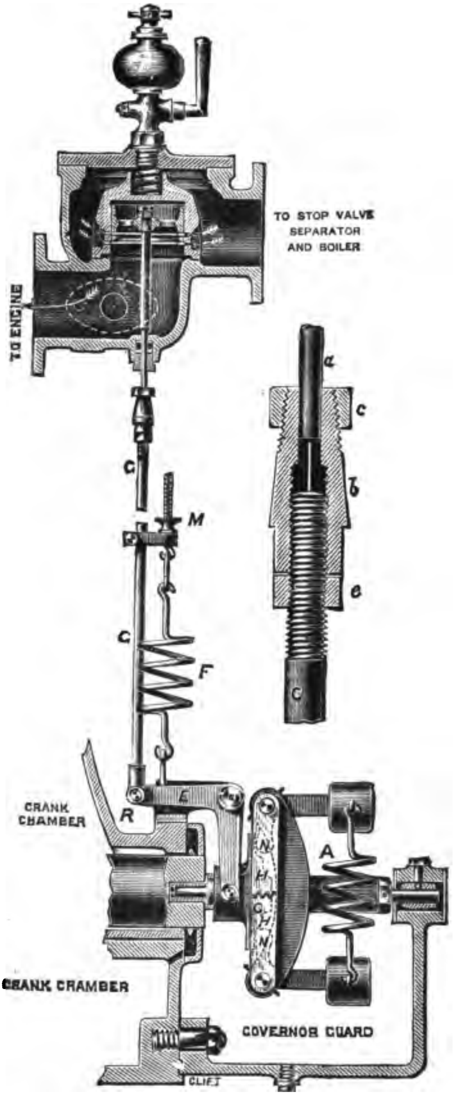


SECTION OF MACFARLANE'S SAFETY GOVERNOR.



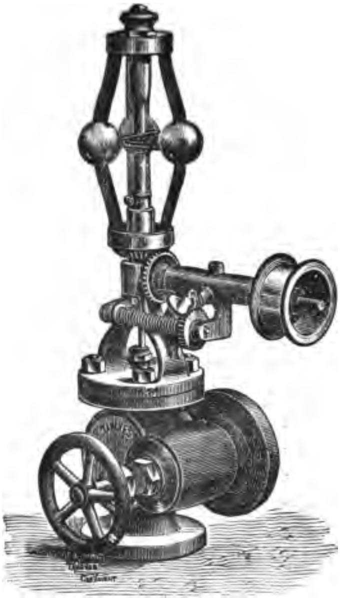
Outside View of Macfarlane's Safety Governor, by Watson, Laidlaw & Co.

governor arms, and thereby causes more friction and wear and tear than would be the case if the springs were directly connected to the balls. In Willans' governor, as will be seen from the figure, the balls are connected directly by a helical spring A, on each side of the governor spindle. Another spring F, is clamped at its upper end to the throttle-valve spindle G, and hooked at its lower end to the bracket carrying the bell crank lever E. This spring pulls the valve rod downwards, in opposition to the springs A, and thus pushes the sleeve against the toes N N (shown dotted), of the governor arms. By adjusting the tension in F, by the nut M, the governor can be set to the required speed while the engine is running. It will be noticed that this governor works horizontally, and is driven directly by one end of the shaft.



WILLANS' SPRING LOADED GOVERNOR.

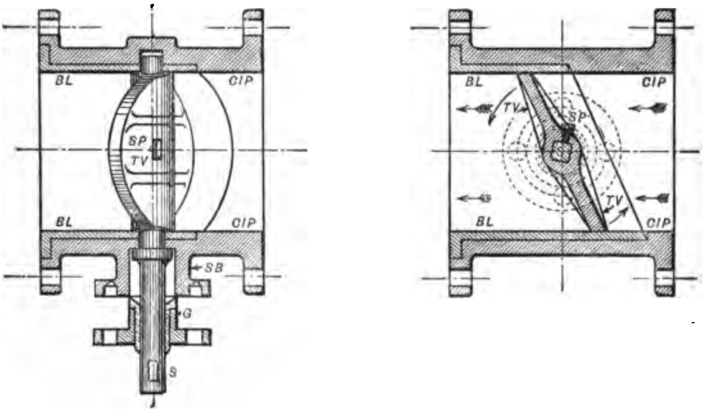
Pickering Governor.—A very simple and direct acting governor which has been introduced for small electric light engines is shown by the next figure. Here the balls are supported by flat springs, which act directly on the throttle-valve spindle. There is also an auxiliary spring, as seen just below the driving spindle, actuated by a thumb screw and worm wheel, which enables the attendant to adjust the speed of the engine whilst running.



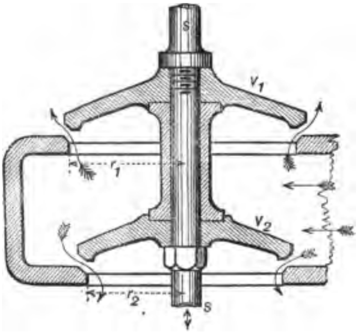
PICKERING GOVERNOR, BY TANGYES, LTD.

Governing by Throttling and Variable Expansion.—Prior to 1876, governors generally controlled steam engines by actuating a butterfly throttle valve of the form shown in the figure. This valve, although simple in construction, is difficult to fit so as to remain steam tight, and hence the double-beat valve as in the next illustration, or still better, a grating piston valve like that shown attached to the Willans' governor, has been adopted in preference. The ordinary butterfly throttle valve is not, as at one time supposed, a balanced valve, since the action of a fluid rushing past an oblique plane, is such as to cause a greater pressure on the forward edge and thus tend to close the valve. A good throttle valve should be able to entirely stop the admission of steam to the cylinder.

Recently, many patents have been taken out for controlling the speed of an engine by altering the point of cut-off. In most cases, this enables the engine to work more economically; but as shown by Captain Sankey in his paper on "Governing of Steam Engines by Throttling and by Variable Expansion," read before the Institute of Mechanical Engineers, in April, 1895, the indicator diagrams obtained from engines governed by this method are often "cloaks for exaggerated initial con-



THROTTLE VALVE.



DOUBLE-BEAT VALVE.

densation," and it may be found that the actual feed water used, is less with ordinary throttling than with variable expansion. Throttling the steam varies the amount supplied by varying the pressure, while the volume used remains constant. On the other hand, automatic expansion supplies the steam at a constant pressure but alters the volume used per stroke.

The point of cut-off may be controlled either by means of a separate expansion valve, or by acting directly on the main valve or valves. In the first case, there are two eccentrics which work the main and the expansion valves. As will be seen from the illustrations of the Hartnell, and Clayton & Shuttleworth's governors, the stroke of the expansion valve is altered by a drag link and a block connected to the governor sleeve. In the second case, when a slide valve is used, either the throw or the position of the main eccentric is varied by a shaft governor, and no second eccentric is required. With "trip gear" the governor automatically releases the admission valves sooner or later, according to the load on the engine.*

Shaft Governors.—A large number of these have been designed, but the following illustrations will serve to show their general principle and action. A circular casting is keyed to the crank shaft, and carries on one side a pair of symmetrically arranged weights jointed thereto at one end, but whose other ends are free to move in a plane perpendicular to the shaft against the resistance of the interposed helical springs. On the other side of this casting there is fixed a pair of straps embracing a circular disc carrying the eccentric which works the valve. The centre of this disc is some distance from the centre of the shaft and that of the eccentric. The governor weights have bosses which pass through slots in the circular casting, and are connected by links to studs on the disc. In moving outwards by centrifugal force, these weights compress the springs and rotate the disc, thus changing the position of the eccentric, and varying the cut-off of the slide valve.

Relays.†—Except in the case of "trip gear," the effort required to work the throttle valve, or expansion gear, may be considerable, and can only be satisfactorily supplied by a relay—that is, by making the engine itself, or steam from the boiler, or water pressure, or electro-magnetic mechanism, move the valve, while

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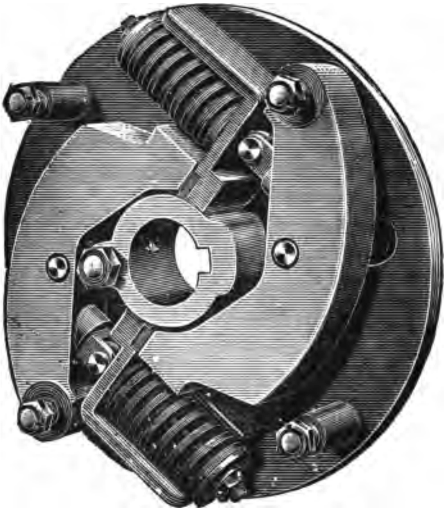
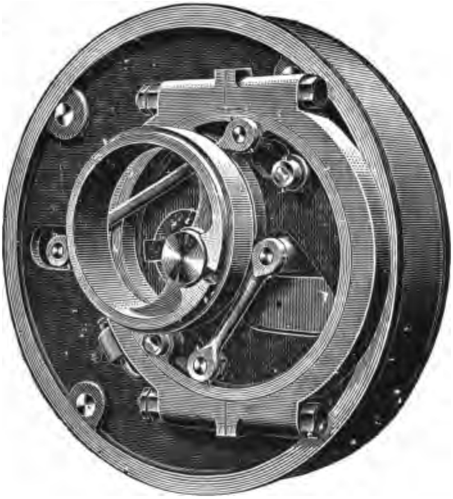
* See Lecture XVIII. of the Author's *Text-Book on Steam and Steam Engines* for illustrations.

† See *Engineering*, 1st January, 1886, p. 4, for a description of Lüdes steam relay governor. Also "Regulation of Steam Engines," by John Richardson, *Proc. Inst. C.E.*, vol. cxx., 1895, Part II., for description and discussion on electrical and other relays for governors.

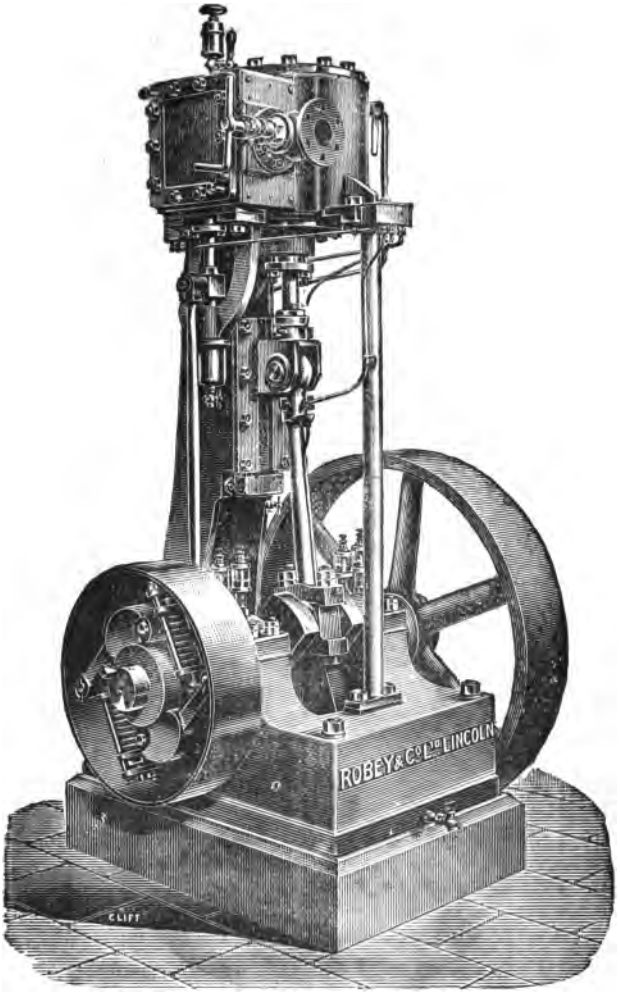


FRONT VIEW.



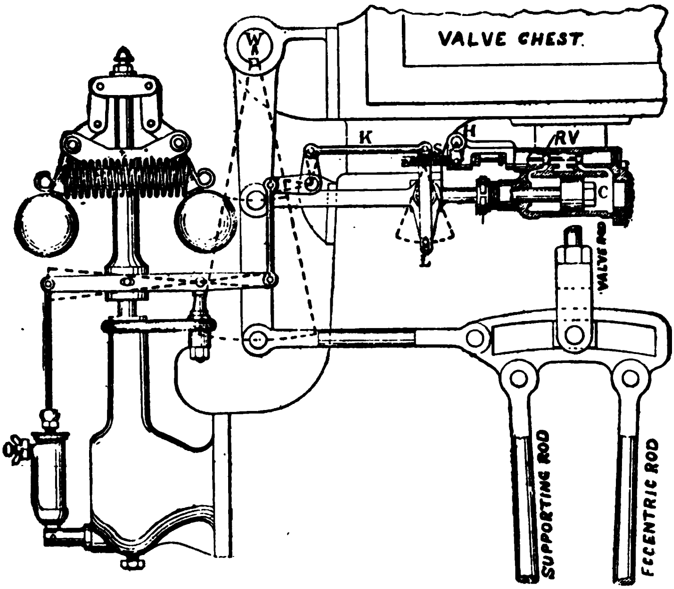
BACK VIEW.

AUTOMATIC EXPANSION SHAFT GOVERNOR,
BY MESSRS. RANSOMS, SIMS & JEFFERIES, LD.



Shaft Governor Applied to a High-speed Electric Light Engine.

the governor has simply to control the relay. In most relays that have been used for governing, the governor starts the relay, and then the latter goes on without any control, until the position of the governor is altered, and it is set into motion in the opposite direction. All such relays necessarily have the fault of hunting, but this is not so for one of the steering-gear type. The governor, as it were, informs such a relay when to move and how far, and the extent of the change in the height of the governor cone determines the travel of the relay.



Governor with Relay for Compound and Triple Expansion Engines, by Davey, Paxman & Co.

The steam relay shown is applied by Messrs. Davey, Paxman & Co. to compound and triple expansion engines. The weigh-shaft W, which works the expansion gears of all the cylinders, is connected with the piston of the small relay cylinder C. The relay valve R V, which admits steam to this cylinder, is worked by the floating lever L, and allows steam to enter at its middle and exhaust at its ends. The lower end of the floating lever is attached at L, to the crosshead of the relay piston rod, and its upper end through the links K, &c., to the governor sleeve, while the small piston valve of the relay is connected to an intermediate point. It will thus be seen that when the governor balls rise, R V will move to the left, and so admit steam to the left of the relay piston. This forces the piston inwards, and pushes the eccentric-rod end of the drag link further from the block attached to the engine valve rod, and therefore reduces the travel of the valve and the power of the engine. In addition, as the relay piston moves one way or the other, it rotates the floating lever L, about its upper end, and brings R V back to its mid position, and so automatically comes to rest. By this means, the relay piston and main slide valves are made to follow all the motions of the governor, and the amount of the motion of the relay piston will depend on the change in the height of the balls. The governor itself has very little work to do, since it has only to move the small valve R V. By means of the weigh shaft W, and levers attached to it the valve rods of all the cylinders are moved simultaneously, in the same way as the one shown. Minor adjustments of the speed may be made while the engine is running by altering the tension in the spring S, by means of a worm and worm wheel on the end of the spindle H.

Knowles' Supplemental Governor.*—Another method is that invented by Knowles. Here two governors are used, a large one to control the valve in the ordinary way, and a smaller one to alter the length of the rod connecting the first one to the valve. This is effected by fitting two friction cones to the sleeve of the smaller, or supplemental, governor, and having a third between them, which will gear with one or other if the governor rises or falls by more than a prescribed amount. The valve rod is in two parts, having a right- and left-handed screw respectively at their adjacent ends, and the nut which joins these screws is rotated by the third friction cone. This governor has been extensively employed in spinning mills, where the fluctuations in load are neither great nor sudden, but where the speed must remain very constant.

Inertia Governors.†—For small gas engines, which always receive a full charge of gas during each cycle or none at all, a form of governor known as the inertia governor, has been found suitable. In the one first illustrated, the gas valve is opened by a valve opener V O, which is actuated through the lever L, by the cam C, fixed on the side shaft S. On the lower end of the valve opener there is a bell crank B C, engaging a slot on the governor weight W. This weight is supported by a spring fixed to a bracket on the lever L. As the lever is moved upwards, the inertia of the weight W, causes it to lag behind, and thus compress the spring, but the latter is so adjusted that as long as the speed does not exceed the normal, B C is not moved down sufficiently to cause V O to miss the gas valve spindle.

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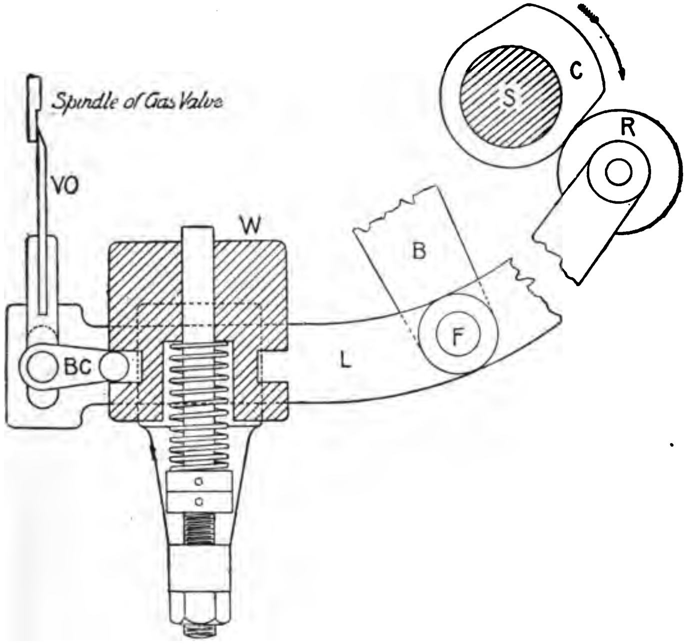
* See the *Practical Engineer*, vol. v., p. 205, March 27, 1891.

† See *Gas, Oil, and Air Engines*, by Bryan Donkin, for other forms of gas engine governors.



Inertia Governor for Stockport Gas Engine.

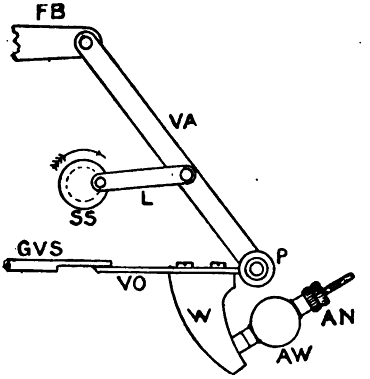
Index to Parts.

C for Cam.
S ,, Side shaft.
R ,, Roller.
L ,, Lever.
F ,, Fulcrum.

B for Bracket.
W ,, Inertia weight.
B C ,, Bell crank.
V O ,, Valve opener.

If, however, the speed should rise above the normal, the inertia of the weight is sufficient to press B C down far enough to cause V O to pass to the right of the spindle, and then no gas is admitted for that cycle. As the direction of the thrust necessary to open the valve passes through the centre of the pin supporting the bell crank B C, it is received direct from the lever L, and does not affect the governor weight W.

Another, and simpler form of inertia governor, as used in small Otto gas engines, depends for its action on the inertia of a small weight W, and adjusting weight A W. The vibrating arm V A, is driven by the link L, from a pin on the end of the side shaft S S, and causes the valve opener V O, and the weight W, to move backwards and forwards. The centre of gravity of the weights and valve opener being to the right of the pin P, the point of V O is pressed against the flat of the gas valve spindle. The effect of the inertia of these weights acts below P, and therefore tends to turn V O downwards when P is moved to the left. The position of A W is so regulated by the adjusting nuts A N, that when the speed exceeds what is desired, the latter tendency will predominate and cause the end of V O to pass below the valve spindle and leave the valve unopened.



Inertia Governor for Otto Gas Engine.

Flywheels.—We have already mentioned that the function of the flywheel is to take up and give out energy so as to minimise the fluctuations in speed due to the periodic changes in the crank effort, and also to reduce the suddenness of other changes in the speed of the engine.

In the previous Lecture we found an expression for the tension per square inch in the rim of a flywheel due to centrifugal force, and saw that it was independent of the cross area of the rim. The highest speed at which a flywheel can be run with safety, will therefore depend upon the tensile strength and the density of the material of which it is made, for it is evident that we cannot make it able to go faster by enlarging the cross area of the rim, since that increases the total stress in exactly the same proportion as it increases the total strength. Consequently, for very high speeds it is necessary to select a material, and so dispose of it, as to have the greatest possible strength for a given mass. Hitherto, flywheels have usually been made of cast iron, either moulded and cast in one piece, or built up in several segments; but recently, they have been made of wrought iron or steel, so as to offer a much greater resistance to bursting, either by the method suggested by Prof. Sharp,* or by winding steel wire into an annular trough made of steel plates at a definite distance from the crank shaft.

Balancing Machinery.†—If a flywheel be not accurately balanced, it will cause wobbling stresses, which produce vibration and wear, and which become greater as the speed is increased. Owing to the recent demand for high-speed machinery, such as sugar-drying, cream separating, hydro-extracting, and electric light machinery, the attention of engineers has been specially directed of late to the necessity for more perfect balancing, with a view to reducing vibration and its attendant noise, tear, and wear. Even in the case of express trains, it has been found advisable to balance the carriage wheels. This is done by placing their axles and their wheels on a framing with springs of exactly the same kind as those to be used on the carriage for which they are intended, and running them at their highest speed of, say, 60 to 70 miles per hour. Pieces of clay are placed upon the inside of their rims until they run perfectly smoothly. These lumps are then replaced by pieces of cast iron or lead of the same weight, and the process repeated until as perfect a balance as possible has been obtained.

In works where the importance of balancing machinery is recognised, the machine to be balanced is placed upon a testing table and run at gradually increasing speeds. At each speed the balance is made as perfect as possible, by trial, in a manner similar to that just described for railway carriage wheels, until the maximum working speed has been reached, and the whole is capable of running practically free from vibration even when not secured by bolts or clamps.

A common method of balancing pulleys in the workshop is to mount them on a shaft, or mandril, which is then placed on two parallel and perfectly level straight edges. This is a delicate method of procuring a *statical* balance, but it does not follow that there is a true *dynamic* balance, as there may be a *centrifugal couple*, which will cause vibration, and needless pressure on the bearings. To take a simple case, consider a crank shaft with two cranks 180° apart. The static balance may be perfect if the cranks are similar; yet, it is clear that the centrifugal forces of the two cranks, although equal, parallel, and opposite in direction, are not in the same straight line, and therefore form a couple in a plane passing through the axis of the shaft. The plane of this couple revolves with the cranks, and it consequently tends to make the axis describe a double cone in space, the common vertex of this cone being at the centre of gravity of the whole rotating mass. Similarly, with a pulley there may be an excess of material on one side at one extremity of a diameter, and at the other extremity an excess on the other side, which, while the static balance is perfect, cause a centrifugal couple, and set up objectionable vibrations at a high speed. The final adjustment of the balance of a wheel or pulley should therefore always be made at the highest speed at which it is intended to run. In order to have a *statical* balance about an axis, it is sufficient that the axis should pass through the centre of gravity of the *whole* mass, but for a perfect *dynamic* balance, it must *also* pass through the centre of gravity of *every* section taken at right angles to the axis. It is possible, however, in some cases to have the body as a whole balanced without this last condition, but in such cases there will be several centrifugal couples whose resultant is zero, but which tend to bend the shaft at several places.

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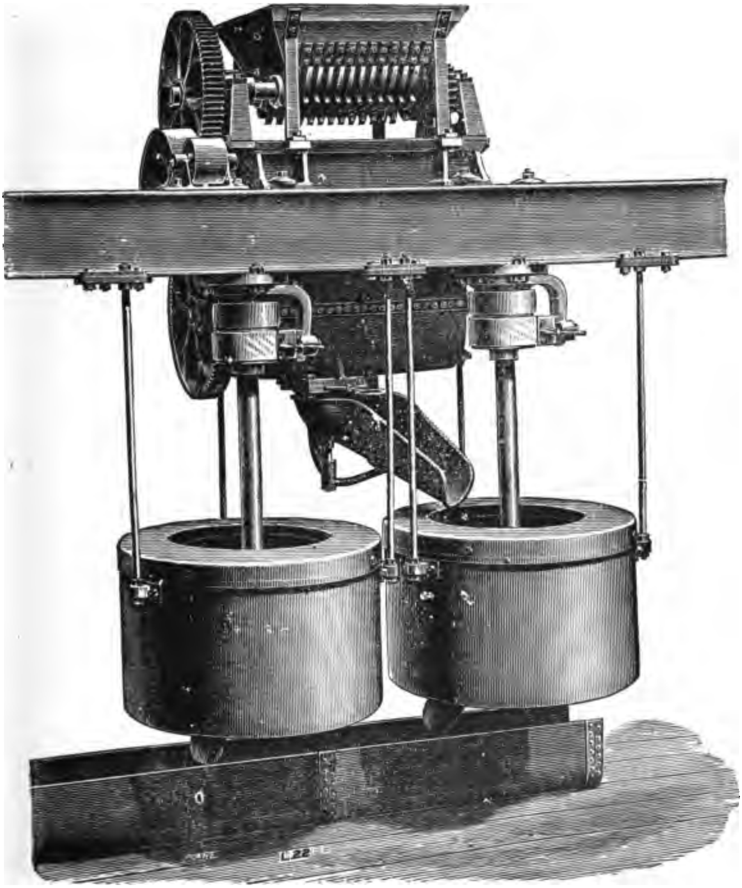
* See Prof. Sharp's pamphlet on "A New System of Wheel Construction" (Technical Publishing Co., Manchester), and a paper on "Flywheels," by John Galt, C.E., M.E., *Proc. Canadian Electrical Assoc.*, Montreal, 1894.

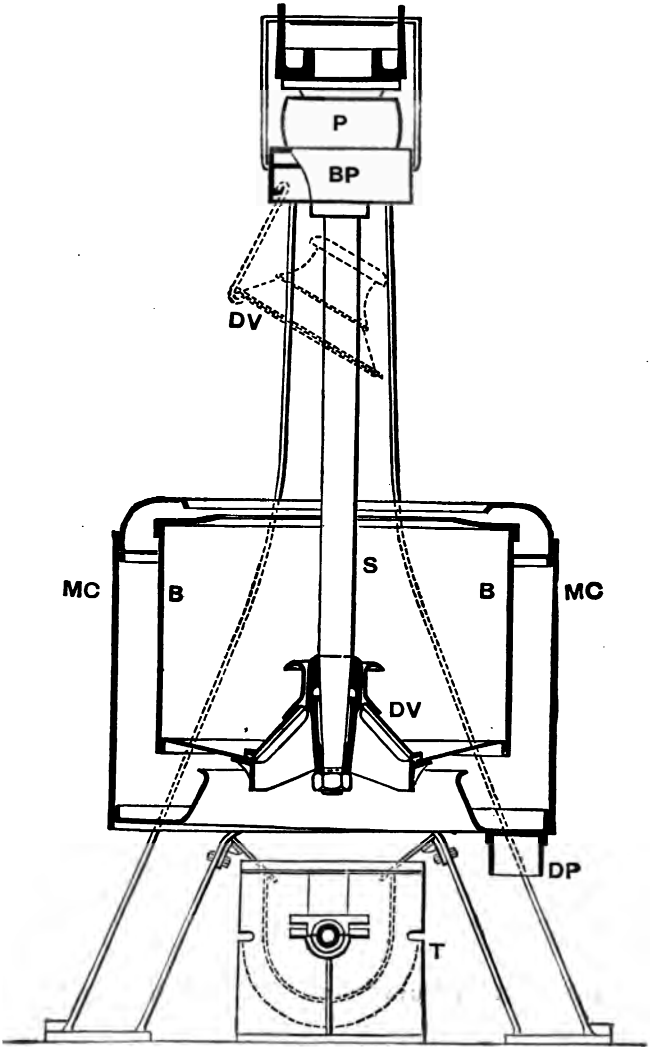
† See *Proc. Inst. Eng. and Shipbuilders in Scotland*, January, 1891, for a paper on "Centrifugal Action in Practical Work," by John Laidlaw. Also, *Proc. N.E. Coast Inst. of Eng. and Shipbuilders*, vol. xii., 1896, for a paper on "An Investigation into the Force tending to produce Vibration in High-Speed Engines," by J. M. Allan.

Weston Centrifugal Machine.—As a useful application of centrifugal force, and an example of a self-balancing high-speed machine, we here illustrate the Weston centrifugal for drying sugar. The first figure gives a general view of a pair of 30-inch centrifugals suspended from the house framing, with sugar-breaker, pug mill, swivel shoot, and molasses gutter. The baskets of these machines are driven at 1,200 revolutions per minute, and give an output of 12 to 16 tons of dried raw sugar, or 12 to 20 tons of dried refined sugar, per day of ten hours, and require about seven horse-power to drive them.

In order to charge the machine, the valve at the bottom of the pug mill is opened, so as to allow the sugar to gravitate down the scoop into the basket B, seen in the vertical section. When a sufficient charge has been given, the pug mill valve is closed, and the basket started rotating by a friction pulley of the kind shown in Lecture VIII., p. 136 of Vol. I. The belt which drives the pulley P, connected to the spindle S, thus gradually brings the speed up to its normal. The centrifugal force causes the water and molasses to pass through the numerous holes in the periphery of the basket into the monitor case M C, from whence it escapes by the discharge pipe D P; while

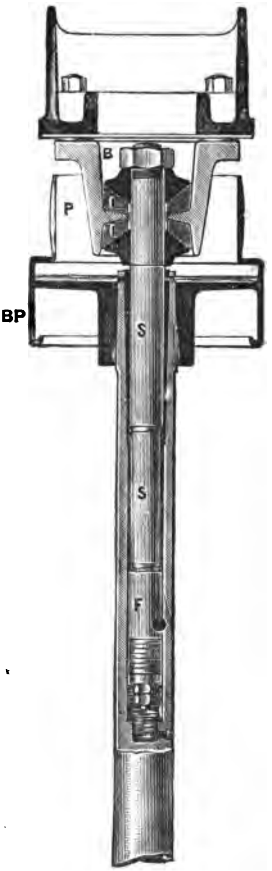


WESTON CENTRIFUGAL SUGAR-DRIERS, BY WATSON, LAIDLAW & CO.



VERTICAL SECTION OF WESTON CENTRIFUGAL.

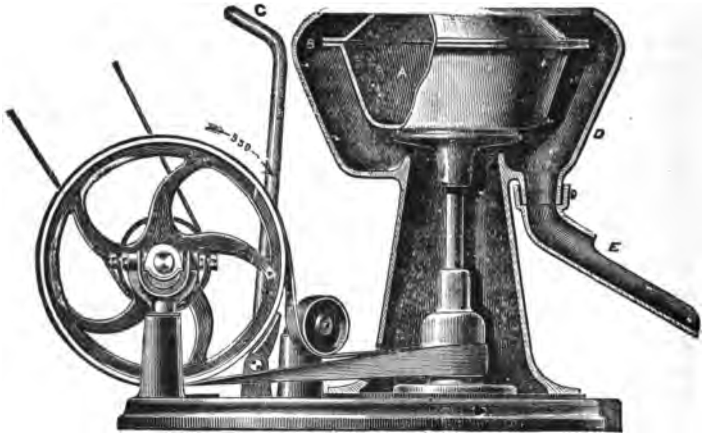
the sugar forms a wall around the inside of the basket. When the sugar is dried, the friction grip of the driving pulley is relieved, and the brake-strap applied to the brake pulley B P, so as to bring the basket and its contents quickly to rest. The conical cover, or discharge valve D V, is then raised and hung on the brake pulley, as seen in dotted lines. The wall of sugar is broken down and swept through the central opening into the conveying trough T. It is then forced along this trough by a large screw to wherever it may be wanted.



SECTION OF SPINDLE AND BEARINGS FOR WESTON CENTRIFUGAL.

The basket is not compelled to revolve about a fixed axis, but is permitted to choose its own centre of rotation by the use of elastic bearings. By allowing the revolving basket to oscillate within certain limits, it assumes as its centre of gyration the centre of gravity of the basket and its contents, and so becomes self-balancing. This reduces to a minimum the power required to drive the machine, severe stresses, wear and tear, and the vibrations transmitted to the building. By referring to the sectional view of the spindle, it will be easily understood how this is accomplished. A strong block B, is bolted to the overhead beam, and inside this block are placed two india-rubber buffers I I, the upper of which sustains the suspended spindle S, by a nut and washer. This spindle does not rotate, but it carries, at its lower end, a series of washers which support the bearing F, fixed to the outer revolving spindle. The pulley P, and brake pulley B P, are attached to the upper end of this outer spindle, and the perforated basket to its lower end. The hollow portion of this spindle is filled with oil, so that the bearing F runs in an oil bath, and is always well lubricated.

There are many other applications of this principle, such as in hydro-extractors, cream-separators, &c. The next illustration shows a modification of the above machine adapted for extracting oil from engine waste, turnings, screws, &c., or drying crystals and ores. The material to be dried is put into the hollow pan A, which is then rotated at about 2,000 revolutions per minute. The oil, or water, escapes through the narrow opening between the upper and lower parts of the pan at B, into the outer casing D, and thence to the spout E. The pan is emptied by lifting it bodily from the top of the spindle and turning it upside down. It rests on a leather-faced disc on the top of the spindle, and is kept central by a continuation of the same, which fits easily into a recess in the bottom of the pan. This arrangement permits of a little slip at starting, by which the driving belt is relieved from any sudden or severe stress. The spindle is similar in construction to the one just described, but inverted, so that this machine is also self-balancing.



UNDER-DRIVEN CENTRIFUGAL EXTRACTOR, BY WATSON, LAIDLAW & CO.

LECTURE XXIII.—QUESTIONS.

1. The governor and flywheel of an engine have both the purpose of regulating its speed. Explain how their actions in this respect differ.

2. Explain the principle of *Watt's* pendulum governor, and state its advantages and defects. Various methods have been proposed for improving this form of governor; discuss the action of any such modern apparatus with which you are acquainted.

3. What are the principal essentials of a good steam engine governor? Sketch, in outline, any one form of governor with which you are acquainted, and explain to what extent it is satisfactory according to the conditions which you have laid down, or how it might be improved.

4. Sketch the ordinary pendulum or ball governor of a steam engine. Mark on your drawing some particular line whose length is related to the number of revolutions of the balls. State the relation as nearly as you know it. If the line referred to be shortened in proportion of 2 : 3. how much would the number of revolutions be increased? *Ans.* $\sqrt{3} : \sqrt{2}.$

5. Sketch an ordinary Watt's governor, and explain its action upon the valve with which it is connected. Why is it an improvement to shift the points of suspension so that the arms cross each other?

6. Explain the advantages of the crossed-arm governor for a steam-engine. Find the height of the cone when the engine is making 40 revolutions per minute, and prove the formula on which you rely. (S. and A. Adv. Steam Exam., 1891.)

7. Define the term "isochronous" as applied to governors. How may isochronism be approximately obtained? Prove the formula, connecting the height of the cone of revolution and the number of revolutions per minute, for a simple pendulum governor. (S. and A. Adv. Steam Exam., 1892.)

8. Sketch the pendulum governor as Watt made it. From the balls of a common governor, whose collective weight is A, there is hung by a pair of links (of lengths equal to the ball-rods) a load, B, capable of sliding up and down the spindle. Compare the loaded and common governor as regards sensitiveness, the weights of the arms or links being neglected. (S. and A. Adv. Steam Exam.)

9. Find an expression for the height of the cone in a loaded governor when rotating at a given number of revolutions per minute. Show, by a sketch, the connection of the governor with a throttle valve. By what arrangement may the tendency to over-sensitiveness be corrected? (S. and A. Hons. Steam Exam., 1891.)

10. Find the height of a simple or "Watt" governor revolving at 80 revolutions per minute. If the same governor had a weight of 40 lbs. attached to the sleeve, the balls weighing 3 lbs. each, what should be its height, supposing the same speed to be maintained, and the link work to be such that the sleeve rises twice as fast as the balls? Neglect the weight of the connecting links. (S. and A. Adv. Steam Exam., 1895.)

11. Find the height of a simple conical pendulum revolving at 80 revolutions per minute. If a loaded governor, making 240 revolutions per minute, had a weight of 20 lbs. attached to the sleeve, the balls weighing 2 lbs. each, what would be its height, the vertical motion of the balls being half that of the sleeve? (S. and A. Adv. Steam Exam., 1894.)

12. Sketch and describe any spring loaded governor, and compare the action of the spring with that of a weight.

13. What objection is there to regulating the speed of an engine by the throttle valve?

14. Explain what is meant by automatic expansion gear, showing wherein lie its special advantages in the economic working of an engine. Sketch such an arrangement and its connections. (S. and A. Hons. Steam Exam., 1894.)

15. Explain by the aid of the necessary sketches the construction of either the Armington-Sims or the Westinghouse high-speed flywheel governor and valve gear. Show clearly how in these arrangements the throw and angle of advance of the eccentric are varied, whilst the lead is kept constant. (S. and A. Hons. Steam Exam., 1895.) (Robey's and Ransoms, Sims & Jefferies' shaft governors are similar to those asked for.)

16. What special benefit is obtained by adding a relay to a governor? Sketch and describe a relay which automatically follows up the motion of the governor.

17. Sketch Knowles' supplemental governor and describe its action.

18. Describe the pendulum governor of the Otto engine, and point out, by reference to sketches, the manner in which it acts. (S. and A. Adv. Steam Exam., 1889.)

19. Explain clearly the arrangement by which the speed of an Otto engine is regulated (S. and A. Adv. Steam Exam., 1891.)

20. Describe any form of inertia governor used for regulating the speed of a gas engine.

21. Describe, with proper sketches, a form of vibrating pendulum regulator as fitted to an Otto gas engine, and explain how it acts, and is made adjustable. Assuming that the pendulum is actuated by the rotation of the gas and air valve, describe the mechanism connecting the end of the valve with the pendulum, showing that it forms a well-known combination in linkwork. (S. and A. Hons. Steam Exam., 1894.)

22. Explain why it is necessary to balance high-speed machinery, and describe the most approved method of doing so.

23. What primary law in mechanics asserts itself when some revolving piece of machinery moves at a high velocity, and is unbalanced? A weight of 1 lb. is placed on the rim of a wheel 2 feet in diameter, which revolves upon its axis and is otherwise balanced. The linear velocity of the rim being 30 feet per second, what is the pull on the axis as caused by the weight of 1 lb.? *Ans.* 28·1 lbs.

24. Explain by sketches and description how railway carriage wheels for express trains and their axles are balanced. Give your reasons for and against the common workshop expression that a perfect statical balance is not one when the machine is run at a high speed.

PART IV.—GRAPHIC STATICS.

LECTURE XXIV.

CONTENTS.—Graphic Statics—A Framed Structure—Classification of Frames—Firm Frames—Deficient Frames—Redundant Frames—Conditions of Equilibrium—Bow's Method of Lettering—Solution of a Triangular Frame—Reciprocal Figure for a Joint—Definition of a Strut—Definition of a Tie—Stress Diagram—Determination of the Kind of Stress in a Bar—Firm Quadrilateral Frame—Firm Triangular Frame—Firm Frame—Firm Frame with Mansard Outline—Questions.

Graphic Statics is the Science and Art of determining by scale drawings the total stresses in the various parts of a structure. The forces transmitted through each part of a structure may be ascertained either by calculation or by graphical construction. The former method is extremely tedious, except in very simple cases, whereas the latter is not only rapid, but also affords a self-evident means of checking the accuracy of the solution.

DEFINITION.—**A Framed Structure consists of an assemblage of rigid bars, so arranged, that the stresses in them are principally push or pull and by the use of which, external forces may be transmitted or modified.**

A structure is different from a machine in so far as, the former transmits force while the latter transmits energy. This means that the parts of a structure are assumed to be at rest while those of a machine must be in motion.

In this section we assume the following, unless otherwise stated :—

(1) That the point of crossing of two or more bars is a joint and perfectly frictionless.

(2) That all the members or bars are able to withstand either push or pull.

(3) That each bar is incapable of being perceptibly deformed under the action of the stress it may have to carry.

For the complete specification of a force, we must know the following four elements :—

(1) The point or place of application.
(2) The line of action—*i.e.*, the line along which the force is acting.
(3) The direction or way the force acts along its line of action.
(4) The magnitude—*i.e.*, the number of units of force.

Classification of Frames.—(1) **Firm Frames** are those which have *just* sufficient bars to prevent change of shape, and any bar may therefore be lengthened or shortened without stressing any of the other members.

(2) **Deficient Frames** are those which have *not* sufficient bars to prevent deformation, and the joints must therefore be made stiff in order to resist change of form.

(3) **Redundant Frames** are those which have *more* bars than are necessary to resist distortion. In frames of this kind we cannot alter the length of certain bars without stressing one or more of the other members. Further, the frame may be self stressed if the redundant bars be badly fitted, and the stresses in the various members are indeterminate unless their yieldingness be taken into account.

Conditions of Equilibrium.—There must be no translation. This is assured if the diagram of external forces is a closed polygon.

There must be no rotation. This is satisfied if:—

(1) The external forces have no resultant moment round any two points that may be chosen.

(2) The line of action of the resultant of all except two of the forces passes through the point of intersection of the lines of action of these two forces.

If a number of external forces act upon a structure and keep it at rest, and, if we have to determine graphically the relations among these external forces, we must know at least:—

Either.—**All the elements of all the forces except one and nothing about that one.**

Or.—**All the elements of all the forces except two, and about one of these two its line of action. About the other, one point in its line of action.**

In the former case, we determine the resultant of all the given external forces by any method, and the last or balancing force (that is, the one we know nothing about) has (1) its point of application anywhere in the line of action of the resultant, (2) its line of action coincident with the line of action of the resultant, (3) its direction or way opposite to that of the resultant, and (4) its magnitude is the same as that of the resultant.

The second case will be clear by a reference to Fig. 1.

B A is the resultant of the external forces, 1.2.3...(n – 2), acting on the body or frame. D C is the line of action of the

$(n-1)^{th}$ force, and E the point chosen as a point in the line of action of the n^{th} force.

If three forces act upon a body and keep it at rest their lines of action must all pass through one point. Thereby, the line of action F E, of the n^{th} force may be determined, since it must pass through O the point of intersection of B A with D C. Then by an application of the triangle of forces the magnitudes and ways or directions of the $(n-1)^{th}$ and n^{th} forces may be determined.

Bow's Method of Lettering.—In Fig. 2 we have an example of Bow's method of lettering a system of forces. It will be seen that every force has one letter on each side of its line of action. This is in order to name the force. Thus, we speak of the forces A B, B C, C D, D E, and E F. Again, each letter has been used twice, excepting A and F. This would indicate that one force was awanting or required to be determined:—viz., the force A F. This force may be the resultant or the equilibrant as the case may be; or, if on drawing the polygon of forces, F coincides with A (that is, the magnitude of F A is zero), then the system is in translationary equilibrium.

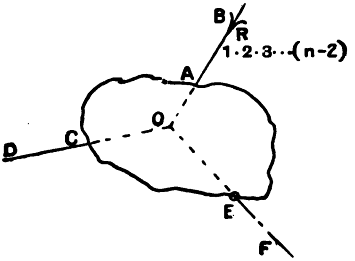


Fig. 1.—Relation among External Forces.

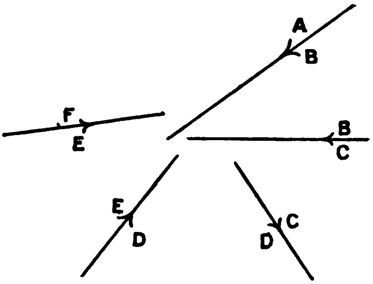


Fig. 2.—Illustration of Bow's Method of Lettering.

In Fig. 3, we have the forces acting at the joints of the triangular frame X Y Z, named by Bow's method.

The forces which keep in equilibrium the joints X, Y, and Z, are:—

For the joint X—

The force B C (all the elements of which are known).
The action of the stress* C D; and,
The action of the stress D B.

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* As is usual, in treatises on this subject, the word *stress*, throughout Part IV., means the *total force* transmitted by the bar, and not the force per unit of cross area.

For the joint Z—

The action of the stress B D;
The action of the stress D A; and,
The supporting force A B.

And for the joint Y—

The action of the stress A D;
The action of the stress D C; and,
The supporting force C A.

The supporting force C A, has been represented by a curved dotted line to indicate that all we know about it is, its point of application.

The point X might be called the joint B C D; the point Y the joint C A D; and the point Z the joint A B D, since the letters naming a joint have been used to name the forces acting at that joint.

In Fig. 3, we have used the letters A, B, and C each four times and the letter D six times. In practice, this is avoided by lettering, as indicated in Fig. 4. Then the forces and bars will have the same names as before. Success in graphic solutions depends in a great measure on correct lettering, and on correct assumptions having been made, first, with regard to the total number of external forces that act on the frame, and second, with regard to what is known about the various elements of these external forces.

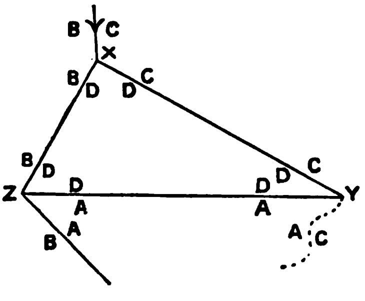


Fig. 3.—Bow's Method of Lettering a Triangular Frame.

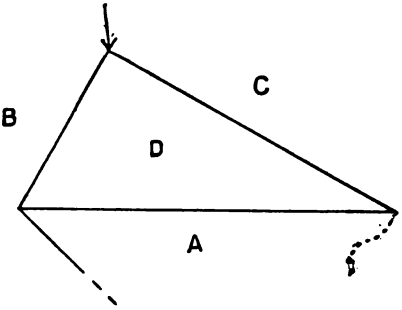


Fig. 4.—Bow's Method in Practice.

One great advantage of the Graphic Method of Solution is, that our attention is always being directed to the correctness of any assumptions that have been made. If the Stress Diagram closes, we may safely consider the solution to be correct for the assumptions made; but, if it does not, some assumption is wrong or something has been left out.

Correct lettering is accomplished when every external force and every bar has one letter and only one on each side of it.

All the external forces must be applied at the joints of the frame, but if any should act at a point other than the end of a bar, then two equivalent parallel forces must be applied to the member under consideration, one at each end. By equivalent parallel forces is meant two forces which, applied as stated, would have the given force as their resultant.

The lines of action of the external forces must not fall inside the frame, but must be drawn outside, as in Fig. 4.

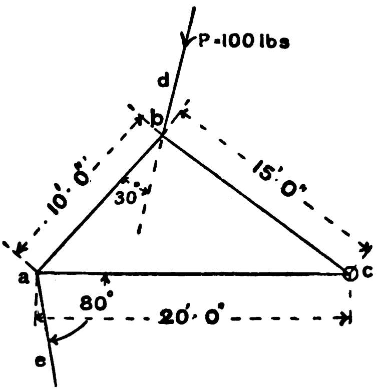


Fig. 5.—Sketch of Frame.

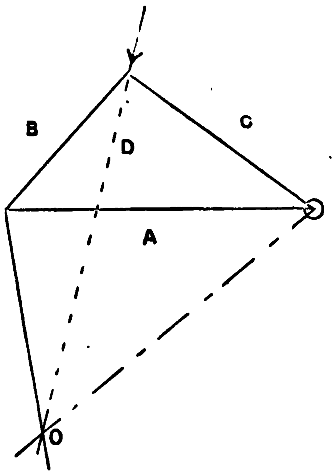


Fig. 6.—Frame Diagram.

Solution of a Triangular Frame.—*Given*, the triangular frame *a b c*, and the force P, completely specified as follows, viz.:—

Its point of application, *b*;
Its line of action, *d b*;
Its way or direction from *d* towards *b*; and,
Its magnitude, P lbs. Also,
The line of action of one of the supporting forces, *a e*.
And finally, a point *c*, in the line of action of the other supporting force.

It is required to find the remaining elements of the supporting forces, also the magnitudes and kind of stresses in the bars.

We begin by drawing the Frame Diagram (Fig. 6) to scale. This scale should be as large as possible, say, in this case, ¼ inch representing 1 foot. Then letter the Frame Diagram by Bow's method. Now, let the lines of action of the forces A B and B C, on being produced meet in O. Then *for no rotation*, the line of action of the other supporting force C A (as indicated by the chain dotted line) must pass through O, and also through the joint D C A, as given. Thus the line of action of the force C A is determined.

For no translation, the triangle of forces is applied, and will give the magnitudes and ways of the supporting forces, as indicated by Fig. 7. The scale used should be as large as convenient, say 1 inch representing 40 lbs.

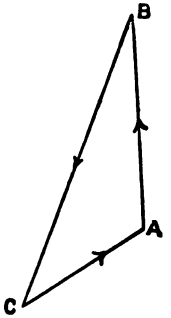


Fig. 7.—Diagram for External Forces.

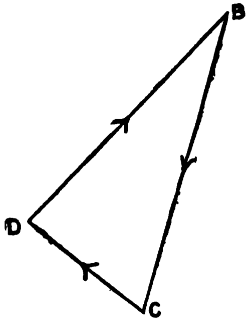


Fig. 8.—Reciprocal Figure for Joint B C D

Definition.—If from a point, a number of lines radiate, and if a polygon be drawn which has its sides either all parallel to, or all at right angles to corresponding radiating lines, then this Polygon is called the Reciprocal of the Point.

Thus, the triangle B C A, Fig. 7, may be called the reciprocal of the point B A C or O, in Fig. 6.

We can now draw the reciprocal figure for any one of the joints of the triangular frame. Because, we know all about the external forces acting at each of these joints; and further, not more than two bars meet at each joint.

If more than two bars meet at a joint, then we must know, in addition to all the external forces acting at that joint, the stresses in all the bars except two, before the reciprocal can be drawn.

Fig. 8 is the reciprocal figure for the joint B C D. It is drawn to the same scale and in exactly the same manner as Fig. 7, viz.:—B C parallel and equal to the external force B C; C D parallel to the bar C D; and D B parallel to the bar D B.

The length of the lines CD and B D, in Fig. 8 (measured to the same scale as that used for B C) determine the magnitudes of two forces which, acting in conjunction with the external force B C, would keep the joint B C D, at rest. The two forces C D and D B, are the actions on the joint of the stresses in the bars C D and D B, and therefore measure the magnitudes of the stresses in these bars. The arrow-heads give the ways or directions along the line of centres of the bars of the actions C D and D B.

Figs. 9 and 10 are drawn to the same scale and in the same way as Fig. 8, and represent the reciprocals for the joints B D A and A D C respectively.

From Figs. 9 and 10 we get similar information regarding the bars A D and D B, and their actions on the joint D B A, and the bars C D and D A, and their actions on the joint A D C, to that derived from Fig. 8 regarding the joint B C D.

In the reciprocal figure for the joint B C D, Fig. 8, the way of the action on the joint B C D, of the stress in the bar C D, is towards the left and upwards, while in Fig. 10 the way of the action of the stress in the same bar on the joint A D C, is towards the right and downwards.

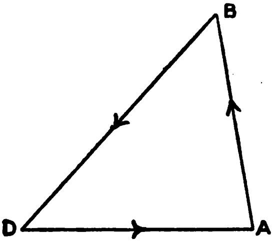


Fig. 9.—Reciprocal for Joint B D A.

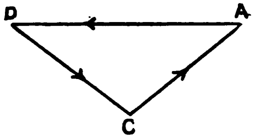


Fig. 10.—Reciprocal for Joint A D C.

We will now explain the cause of this apparent contradiction in the two reciprocals. The reciprocal for the joint B C D, shows that the way of the action on the joint B C D, of the stress in the bar D C, is towards the pin B C D—that is, pushing it. (This is indicated in Fig. 11 by the small arrow.) Then from Newton's third law the pin B C D, must push the bar with an equal and opposite force.

If the bar D C, pushes the pin B C D, it must also push the pin C A D. For this reason, that no bar can simultaneously push a pin at one end of itself and pull one at its other end. This is what the reciprocal for the joint A D C, indicates.

Definition of a Strut.—**When the reciprocal, for a joint indicates that the way of the action of the stress in a bar is towards the joint, then that bar is under compression and is called a strut.**

On reference to the reciprocals for the joints B D A and A D C, it will be seen that the bar D A, is pulling at the pins B D A and A D C. But, by the action and reaction law, the pins will pull at the ends of the bar, and this means that the bar D A, is under tensional stress of an amount measured by the length of the line D A, in the reciprocal figures.

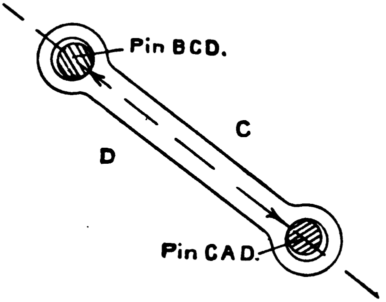


Fig. 11.—The Bar C D.

Definition of a Tie.—**When the reciprocal for a joint indicates that the way of the action of the stress in a bar is away from the joint, then that bar is under tension, and is called a tie.**

In Fig. 12, the reciprocals for the three joints of the frame and the one for the point O, have been combined into one diagram, which may be called either the Stress Diagram or the reciprocal of the Frame Diagram.

Definition.—**Two figures are reciprocal when every point in the one has a corresponding reciprocal in the other.**

For example, the point C, in Fig. 12 has the lines D C, A C, and B C, meeting in it. If we refer to the Frame Diagram, Fig. 6, we find that, the bar C D, the force B C, and the force A C, form the reciprocal for this point C in Fig. 12.

We do not put arrow heads on the Stress Diagrams; they would lead to confusion, and are quite unneces-

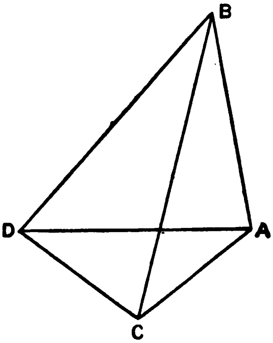


Fig. 12.—Combination of the Reciprocals, or Stress Diagram.

sary. Take for example the line A D, Fig. 12, from the reciprocal of the joint A B D, we would require an arrow head pointing from D towards A; while, from the reciprocal of the joint A D C, an arrow head would require to point the other way. (The double arrow on the line A D, points to the fact that a stress has no way.)

It will have been observed:—

(1) That when Figs. 7, 8, and 9 have been drawn they give all the information that was intended to be derived from drawing Fig. 10—viz., Fig. 8 gave the magnitude of O D, and Fig. 9 that of D A.

(2) That when we place the reciprocal for the joint A B D, on the reciprocal for the point O, as in Fig. 12, we have only to join D to O in order to complete the diagram.

These two observations point out that we have too much information; the excess is due to the finding of the point O. This frame is one of a class where we may find the stresses without first finding all the elements of the reactions or supporting forces.

Stress Diagram.—We shall now show how to determine the Stress Diagram direct from the Frame Diagram—*i.e.*, without first finding the reciprocals for the joints, and combining them into one.

It is quite immaterial as to which way we go round a structure —*i.e.* (referring to Fig. 6), whether we go from A to B and then to C, or the other way round. We shall find it to be an advantage to go round every structure in the same way as the hands of a watch. By doing so we shall find that the Stress Diagram will always lie to the left hand of the external force polygon. This will enable us to know where to begin the external force polygon in order to leave room for the Stress Diagram.

Referring to Fig. 6, where we are not supposed to know either the point O, or the line of action of the force C A, let us plot out therefrom the Stress Diagram, Fig. 12.

(1) Draw B C, in Fig. 12, parallel to the line of action of the external force B C in the Frame Diagram, Fig. 6, and containing 100 units, to some convenient scale, say 1 inch to represent 40 lbs. The correct lettering of this line is a very important part of the work. Since we are going round the frame in the direction of the hands of a watch—that is, from B to C—then B must be put at the top end and C at the bottom end of the line just drawn so as to indicate the way of the force correctly. If this point is attended to, little trouble will be experienced in drawing the diagrams.

(2) Draw from the last point found (viz., C) a line parallel to some force or bar which has C as one of the letters for its name; for example C D.

(3) Draw from the other end B of the line B C, a line parallel to the bar B D, and mark the point of crossing of the two lines D.

(4) Through D, the last point determined, draw D A parallel to the bar D A, and from some of the other points found draw a line parallel to a force whose line of action is known. Now, C A cannot be used because we only know its point of application, but we know the line ot action of B A. Then drawing from B, in Fig. 12, a line parallel to the line of action of the supporting force A B, we determine the point A.

(5) On joining C with A we obtain a line parallel to the line of action of the supporting force C A, and the length of this line, C A, measures the magnitude of the force.

(6) Measuring the lines in Fig. 12 with the scale used to draw down the line B C, we obtain the magnitudes of the stresses in all the bars and of the two supporting forces.

How to Determine the kind of Stress in a Bar.—We will begin with the consideration of the forces which act on the left-hand joint—viz., the joint B D A, in Fig. 12.

Success in this part of the work depends almost entirely upon giving to each bar meeting in the joint under consideration its proper name—*i.e.*, by letters in their proper order.

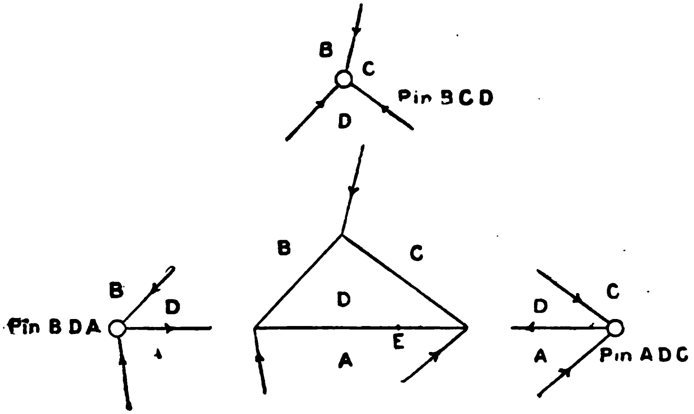
Since we have gone round the external forces in drawing the Stress Diagram from B to C, &c.—that is, in the direction of the hands of a watch—we must go round each joint of the structure in the same way when naming the bars meeting in that joint.

The bars meeting in the joint B D A, would therefore be called, the bar B D, the bar D A, and the supporting force or reaction A B. Having thus determined the name of the bar, we then refer to the Stress Diagram in order to find the way in which the stress in that bar acts with regard to the joint.

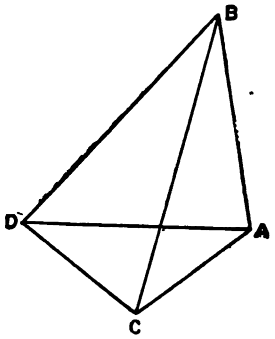
Take for example the horizontal member in the Frame Diagram, Fig. 13 (this member is called the tie rod or tie beam, since it ties the lower ends of the rafters together), its name with reference to the joint B D A, is D A. Now, in the Stress Diagram D is on the left of A, and, therefore, the stress in the bar D A, acts from left to right (*i.e.*, from D to A) with respect to the pin at the joint B D A. This means that the bar D A, is pulling at the pin B D A, and therefore the pin B D A, pulls at the bar, thereby putting the bar into tension.

Similarly the stress in the bar B D (called a rafter) acts, so far as the joint B D A is concerned, in the direction indicated by B D in the Stress Diagram—that is, from B to D. The bar B D, is therefore pushing at the joint B D A, and is thus put into compression by the reaction of the pin B D A.

RULE TO DETERMINE THE KIND OF STRESS IN A BAR.—**Take** the letters on each side of the Bar in the Frame Diagram, in the same order with respect to the joint on which the Bar acts, as we have taken the letters on each side of the External Forces acting on the Frame. Then along the line in the Stress Diagram, which is named after the Bar, from the first letter towards the second, gives the way of the stress' action with respect to the joint under consideration. If the way is towards the joint the stress in the Bar is Compression or Push, and the Bar is called a Strut. If the way is away from the joint the stress is a Tension or Pull, and the Bar is called a Tie.



Frame Diagram.



Stress Diagram.

Fig. 13.—Stress Action on Pins.

The above rule may also be applied to a point in a bar. Take, for example, the point E, in the tie rod, and suppose we want to find how the left-hand portion of the tie rod acts upon the section at E. Then the name of the left-hand portion of the tie rod with respect to E is A D, and from the Stress Diagram this acts from right to left—that is, away from E—and is therefore pulling at the section.

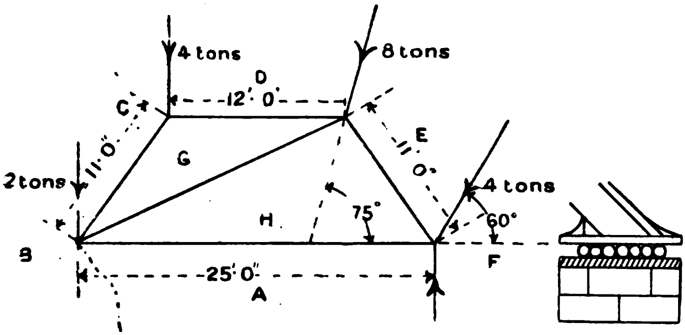


Fig. 14*a*.—Frame Diagram.

Frame Solvable without knowing all about Reactions.

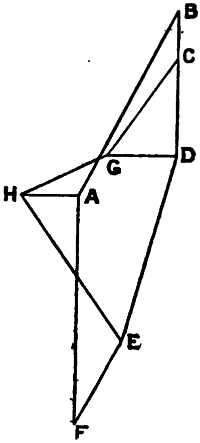


Fig. 14*b*.—Stress Diagram.

Notice that the tie rod, with respect to the left-hand joint, is called D A, and with respect to the right-hand joint would be called A D, and similarly with any other bar in the frame.

The above rule for the kind of stress does away with the use of arrow-heads and of supplementary diagrams.

The action of all the bars on the pins of the frame are shown in the small diagrams surrounding the Frame Diagram of Fig. 13.

Firm Quadrilateral Frame.—This frame is one of a type which allows a solution to be found without having first determined all the elements of the reactions.

We shall assume that the right-hand end rests on rollers, as indicated in Fig. 14*a*. Consequently the line of action of the reaction is practically vertical. If it simply slides instead of rolling, then the reaction is inclined to the normal at an angle equal to the angle of friction, and inclined to that side of the normal which will oppose the motion of the frame. The left-hand end of the frame in Fig. 14*a* is assumed to be anchored to the wall by bolts, &c. All we know about the left-hand reaction is its point of application.

If we suppose both ends of a frame to be anchored, then we only know the points of application of the reactions, and we assume that their lines of action are parallel to each other and to the line of action of the resultant of the external forces.

We begin by drawing the Frame Diagram to as large a scale as possible, and then indicate the external forces at the joints by their lines of action. The right-hand reaction is indicated by a vertical line, and the left-hand reaction by a dotted curved line, as shown in Fig. 14*a*. We then letter the diagram according to Bow's method.

In drawing the Stress Diagrams, we shall always go round the Frame Diagrams clockways.

We begin by drawing a line parallel to the line of action of the first force or load B C. This line should contain as many units of length as B C contains units of force, which in this example is 2 tons.*

Then draw C D parallel to the line of action of the load C D, and containing 4 units of length corresponding to the 4 tons load. Next draw D E parallel to the line of action of the load D E, and E F parallel to the line of action of the load E F, representing 8 units and 4 units, respectively.

The line B C D E F is called the Line of Loads.

In order to complete the Stress Diagram we shall begin with the joint C D G, which is the only joint of which we have sufficient data. Draw from the point C in the "Line of Loads" a line parallel to the bar C G, and from D a line parallel to the bar D G. The intersection of these two lines is called the point G. From G draw G H parallel to the bar G H, and from E draw E H parallel to the bar E H. This determines the point H. Then draw H A parallel to the bar H A, and from F draw a line parallel to the line of action of the reaction F A. The intersection of these two lines fixes the point A. Joining A with B gives the finishing line of the Stress Diagram. The line A B in the Stress Diagram is parallel to the line of action of the left-hand reaction.

By applying the rule for the kind of stress, we can determine from the diagram all we may wish to know—*e.g.*, with respect to the top right-hand joint, the diagonal member is called H G.

---

* The scale for the diagram should be as large as convenient. A rough guess may be made by adding all the loads together, and assuming that this will be the total vertical length of the diagram.

On reference to the Stress Diagram we see that the way of its action is from H to G which means pushing at the joint. Therefore, the diagonal member is in compression, and so on for the other members.

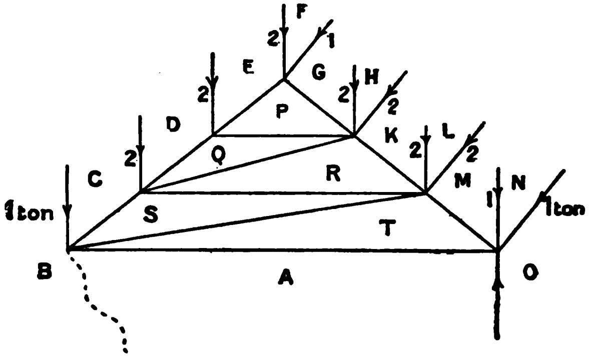


FIG. 15a.—FRAME DIAGRAM.
FRAME WITH WIND PRESSURE.

The magnitudes of the stresses are measured by the lengths of the lines in the Stress Diagram.

The polygon B C D E F A is called the polygon of external forces.

Firm Triangular Frame.—This frame, Fig. 15*a*, can also be solved without knowing all about the reactions.

The right-hand end of the frame is assumed to be resting on rollers, while the left-hand end is anchored to the wall. The vertical loads on the Frame Diagram represent the action of gravity on the roofing, such as slates, &c., which is assumed to be uniformly distributed over the surface.

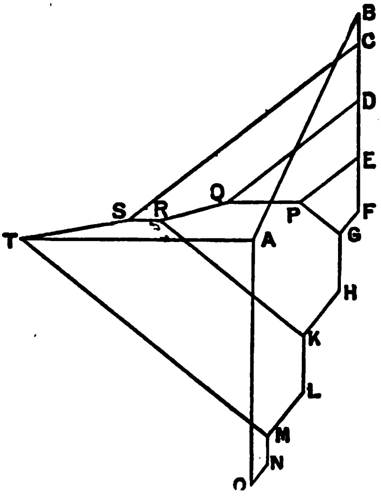


FIG. 15b.—STRESS DIAGRAM.

In Fig. 15*a*, the rafters are shown divided into three equal parts called bays; and, since the joint at each end of a bay must carry one half of the uniformly distributed load over that bay,

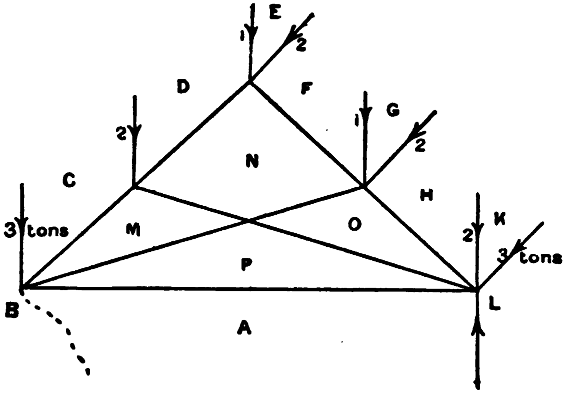


FIG. 16*a*.—FRAME DIAGRAM.
FIRM FRAME WITH A QUADRILATERAL PART.

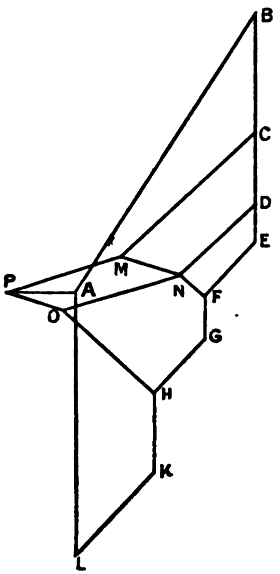


FIG. 16*b*.—STRESS DIAGRAM.

the vertical loads will have the proportions shown by the numbers on the Frame Diagram. Wind pressure is also assumed to be uniformly distributed, and is reckoned as so many lbs. per square foot normal to the rafters. This is indicated on the right-hand side of the Frame Diagram.

Note.*—When wind pressure acts on the rafter which is anchored, the stresses in the members of the frame are more severe than when it acts on the free rafter. This should be remembered when designing a roof.*

Since we know all the elements of the external forces, the line of loads may be drawn as in Fig. 14*b*.

Therefore, in order to complete the Stress Diagram we can begin at the top joint of the Frame Diagram where only two members meet. This enables us to find first the point P in the Stress Diagram, then the point Q, and so on.

Firm Frame.—The frame represented in Fig. 16*a* is of the same class as the two preceding. In drawing the Stress Diagram, although we have determined the point N and the point M, we cannot fix the point P, until we obtain the point O. After that, the diagram closes in the usual way.

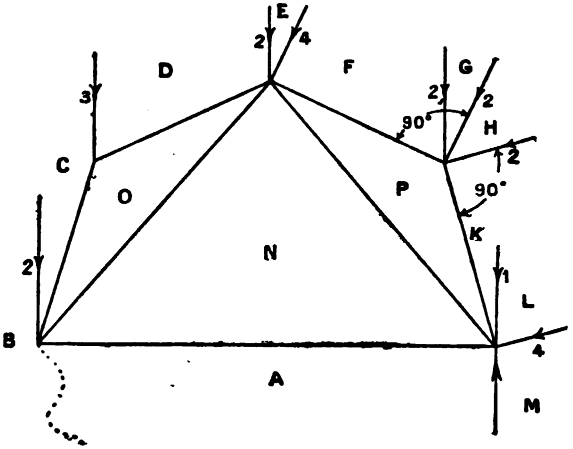


FIG. 17*a*.—FRAME DIAGRAM.

FIRM FRAME WITH MANSARD OUTLINE.

Firm Frame with Mansard Outline. — In Fig. 17*a* we have illustrated a frame having the double-sloped outline of the Mansard Roof. It is of the same type as Fig. 16*a*, and presents the same peculiarity in the drawing of the Stress Diagram. Wind pressure is indicated on the right-hand rafters. The forces E F and G H are both normal to the bar F P. *They should, however, be of equal value.* Also, the forces H K and L M are both perpendicular to the bar K P.

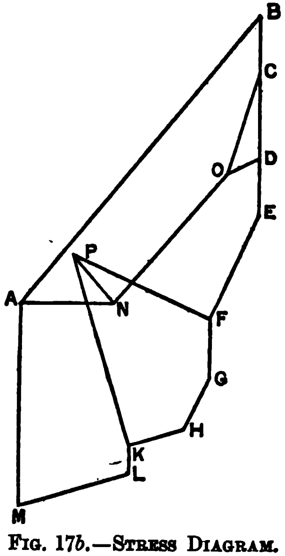


FIG. 17*b*.—STRESS DIAGRAM.

LECTURE XXIV.—QUESTIONS.

1. What is a frame or framed structure? Distinguish between the three different kinds of frames.
2. Explain in your own words Bow's method of lettering a system of forces, with two examples.
3. What is meant by the reciprocal of a point, and a pair of reciprocal figures?
4. Explain how you would represent forces in a diagram so as to determine those in each part of a structure, and explain the principles upon which the construction depends.
5. State a rule for determining the kind of stress in a bar.
6. Illustrate and explain how you would find the stresses in a firm quadrilateral frame.
7. Illustrate and explain how you would find the stresses in a firm triangular frame.
8. Illustrate and explain how you would find the stresses in the outline of a Mansard frame.

LECTURE XXV.

CONTENTS.—Substituted Frames—King Post Truss—Right-Angled Strut Truss—Roof Truss—Load at an Internal Joint of a Frame—Modified French Truss—Bowstring Truss—Questions.

Substituted Frames.—The types of frames illustrated in the previous Lecture, although not practical examples, are intended to be substituted for some other actual form in order to determine the reactions therein, since **the reactions do not depend upon the form of frame carrying the roofing, but merely on the distribution of the loads.** In substituting one of the above frames for a practical one, we must have the joints of the substituted frame coincident with those of the given one. This will be illustrated by the following examples :—

King-Post Truss.—In Fig. 18 we have the Frame Diagram of a king post truss with wind pressure on the right-hand rafter. In this case, we assume both rafters to be anchored to the walls. Therefore, all we know about the elements of the reactions are their points of application, and that their lines of action are parallel to each other, as well as to the line of action of the resultant of the external forces.

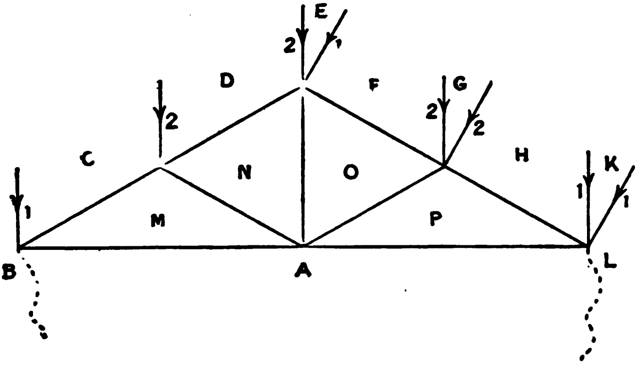


FIG. 18.—FRAME DIAGRAM OF KING POST TRUSS.

Before we can determine the Stress Diagram for this frame we must first determine the reactions, because more than two bars meet in each of the joints except the two where the reactions act. Consequently, until we determine all the elements of the reactions we cannot begin at either of these two joints.

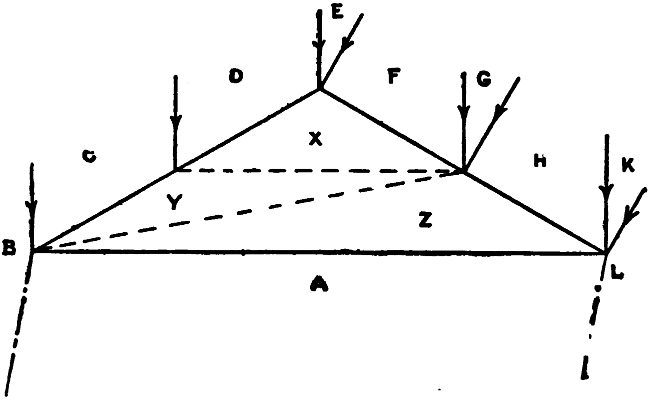


Fig. 19.—Substituted Frame.

In order to determine the reactions, we shall substitute a frame similar to that illustrated in Fig. 15*a*. This substituted

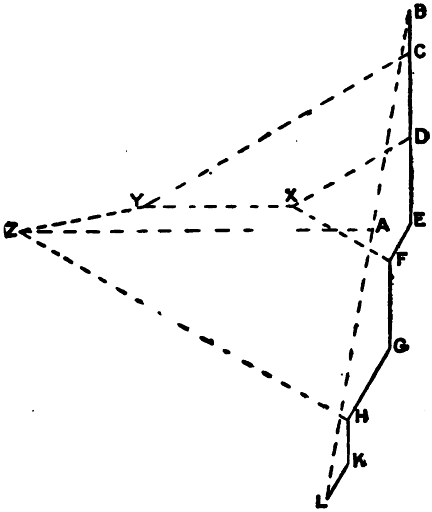


Fig. 20.—Stress Diagram for Substituted Frame.

frame is shown in Fig. 19. In practice, this frame is merely sketched in order to apply the proper letters. The dotted lines are drawn in the Frame Diagram, or the set square is simply made to pass through the requisite joints, and then the lines are drawn parallel thereto in the Stress Diagram. To obtain Fig. 20 we begin by drawing the line of loads. Then, we find the point X, when a line from the point X drawn parallel to the substituted bar X Y, and one from the point C parallel to the rafter C Y fix the point Y. Next we find the point Z. Now, the line of action of the resultant of the external forces is

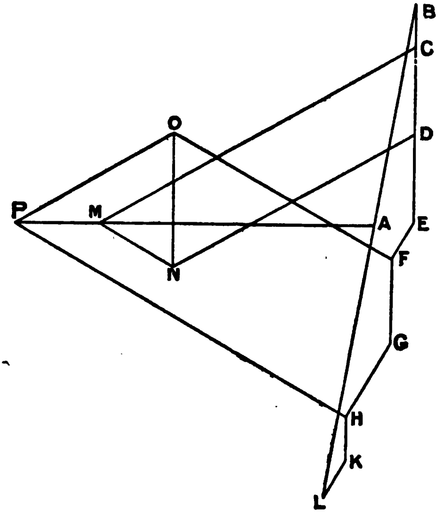


FIG. 21.—STRESS DIAGRAM FOR KING POST TRUSS.

parallel to the line joining L with B. Therefore, the point A must lie on this line since the reactions L A and A B are parallel to each other and to the line of action of this resultant. Consequently, we find the point A by drawing through the point Z a line parallel to the bar Z A so as to intersect L B in the point A. This determines all the remaining elements of the reactions, viz.:—

(1) Their lines of action parallel to L A and A B.

(2) Their ways from L towards A, and from A towards B.

(3) Their magnitudes by the number of units of length in the lines L A and A B.

We can now draw the Stress Diagram for the king post truss, as shown in Fig. 21, from which the particulars for the various members may be determined. Comparing Fig. 21 with Fig. 20, we see that nearly all the lines of Fig. 21 lie along the lines of Fig. 20. In practice we simply draw Fig. 21 on the top of Fig. 20.

This method of a substituted frame introduces fewer errors due to drawing, than the usual method of the funicular polygon (which will be illustrated further on), because we make use of the same joints of the frame for the two Figs. 20 and 21, and the same line of loads.

There is one line in Fig. 21 which will check the accuracy of the Stress Diagram. In drawing the diagram we begin with the point M, and then find the points N, O, and P. The line joining P to H will then be parallel to the rafter P H, if the Stress Diagram is correct.

Right-Angled Strut Truss.—In this frame we have introduced loads at the lower joints as well as roofing weights and wind pressure. We also assume the two rafters to be anchored to the walls, as indicated by the two dotted curved lines L M and A B.

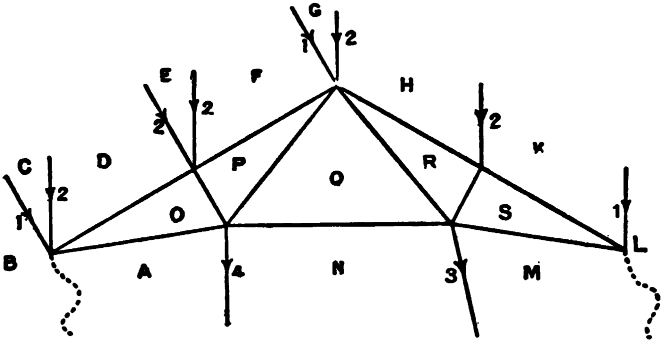


FIG. 22.—FRAME DIAGRAM FOR RIGHT-ANGLED STRUT TRUSS.

We must first find the reactions before we can draw the Stress Diagram. In finding the reactions we will substitute the frame which is shown in Fig. 23.

FIRST METHOD OF OBTAINING STRESS DIAGRAM FOR ORIGINAL FRAME.—Produce the lines of action of the loads at the lower joints until they intersect the rafters. These points of intersection are considered as joints in arranging the substituted frame and the lower loads assumed to be acting at these joints as shown in Fig. 23. The forces *f* F and H *h* in Fig. 23 are the loads N A and M N in Fig. 22 transferred as explained.

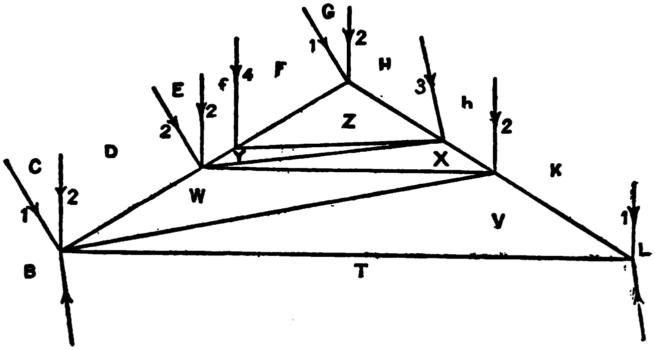


FIG. 23.—SUBSTITUTED FRAME.

The Stress Diagram for the substituted frame is illustrated in Fig. 24 and presents no difficulty requiring explanation. This diagram gives the reactions L T and T B.

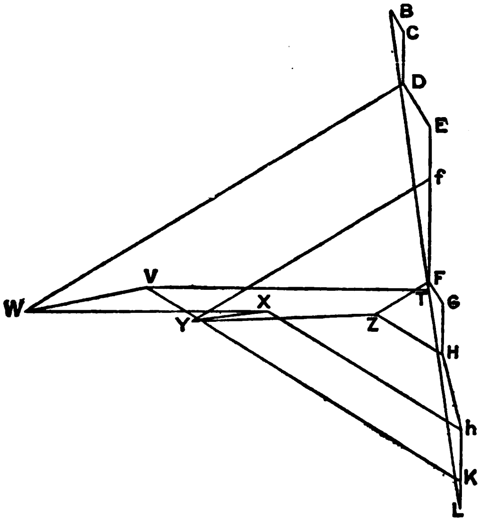


FIG. 24.—STRESS DIAGRAM FOR SUBSTITUTED FRAME.

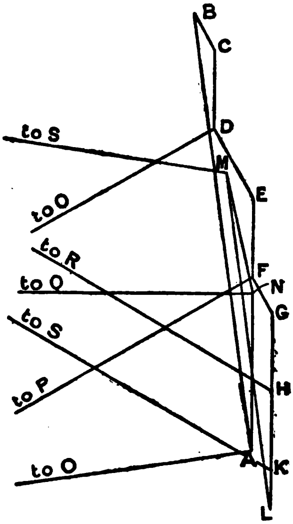


FIG. 25.—STRESS DIAGRAM FOR ORIGINAL FRAME.

In order to draw the Stress Diagram for the original frame which is illustrated in Fig. 25, it is necessary to redraw the line of loads taking them in their order as in Fig. 22. That is, B C, C D, D E, E F, F G, G H, H K, K L, reaction L M, M N, N A and then reaction A B. The drawing of the remaining part of the Stress Diagram calls for no special remark. (The above Stress Diagram is not completed for want of space.)

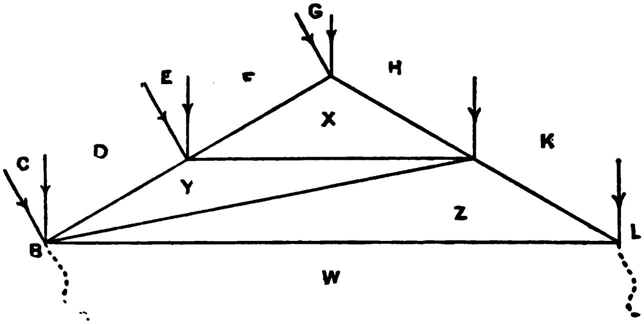


FIG. 26*a*.—FRAME DIAGRAM.
SUBSTITUTED FRAME FOR TOP JOINT LOADS.

SECOND METHOD.—Take the top joint and the lower joint loads separately. In Fig. 26 we have the substituted frame for the top joint loads and its Stress Diagram. The Stress Diagram, Fig. 26*b*, determines the reactions L W and W B due to the loading on the rafters.

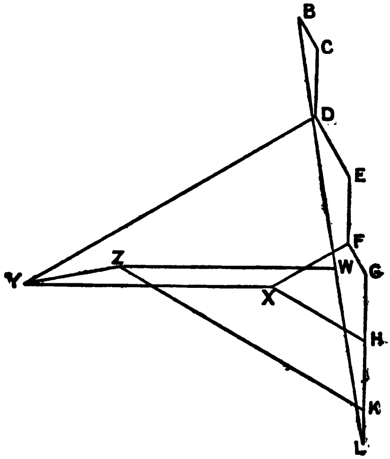


FIG. 26*b*.—STRESS DIAGRAM.

In Fig. 27 we have a frame similar to the one illustrated in Fig. 14*a*. The joint A N S is any point in the line of action of the load N A in Fig. 22, and the joint N M R S any point in the line of action of the load M N in Fig. 22. The left and right hand lower joints of Fig. 27 are the rafter ends in Fig. 22

In Fig. 27 the load T A is the reaction W B found in Fig. 26*b* reversed. The loads A N and N M are the loads at the lower joints in Fig. 22, and the load M V is the reaction L W of Fig. 26*b* reversed. Therefore, if we draw the Stress Diagram for the frame of Fig. 27 we determine the reactions due to all the loads of the original frame of Fig. 22.

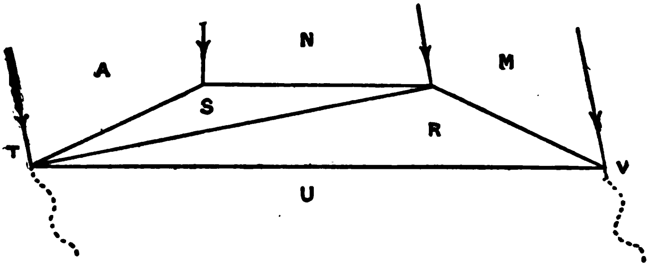


FIG. 27.—SUBSTITUTED FRAME DIAGRAM FOR LOWER JOINT LOADS.

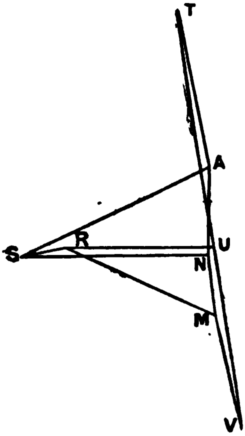


FIG. 28. — STRESS DIAGRAM FOR FRAME IN FIG. 27.

In Fig. 28 we have the Stress Diagram for the frame of Fig. 27. The reactions are represented by the lines V U and U T. This figure has been drawn to a smaller scale than Fig. 24, but the lines V U and U T of Fig. 28 contain the same number of units as L T and T B of Fig. 24.

Roof Truss.—In Fig. 29 we have a frame of a type that will not allow of the Stress Diagram being drawn in a regular manner, but only in a step by step process.

Fig. 30 shows the substituted frame used in order to determine the reactions L A and A B, and in Fig. 31 we have the Stress Diagram for both Figs. 29 and 30.

In Fig. 31 we draw first the line of loads, second we find the point O, then O X and D X fix the point X, and X Y and K Y fix the point Y. On drawing Y A parallel to the bar Y A, and L A parallel to the line of action of the reaction at the right-hand joint, we determine the reactions L A and A B.

The point O is the same in both Stress Diagrams, but we cannot determine the points N and Q until we have found the points M and R. The point M is found by drawing D M parallel to the rafter D M, and A M parallel to the tie-bar A M; then M N and N O fix the point N, and similarly for the point Q. Again, although N may be determined, P cannot be fixed until we have found Q.

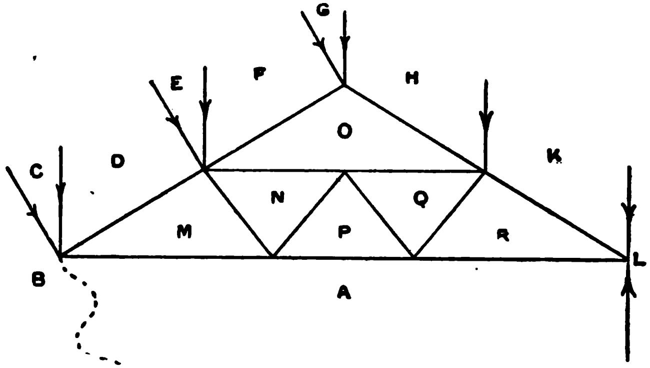


FIG. 29.—FRAME DIAGRAM OF ROOF TRUSS.

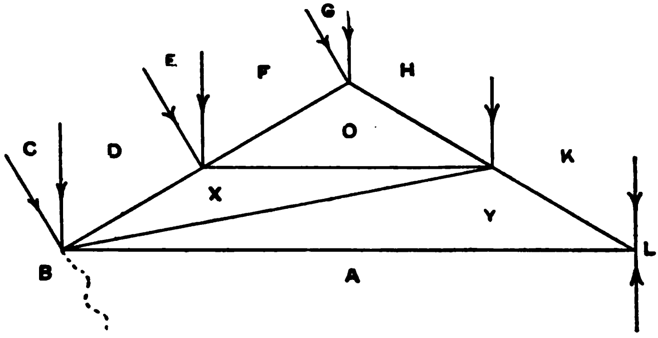


FIG. 30.—SUBSTITUTED FRAME FOR FIG. 29.

The line N P or P Q forms a check line.

***Note.**—The dotted lines of Fig.* 31 *are the only part of the Stress Diagram for the substituted frame that has not been required for the original frame.*

Load at an Internal Joint of a Frame.—In Fig. 32*a* we have a frame the same as the last one, but with a load applied at a joint inside the frame. This load is represented in the Frame Diagram by the small arrow near the letter P, and is applied at the joint O P Q T.

Note.—If the force had been applied at a point in the bar, then equivalent parallel forces *applied to the joints at the end of the bar will allow a solution to be determined.*

To obtain a solution for this frame loaded as shown, introduce a bar with its centre line lying along the line of action of

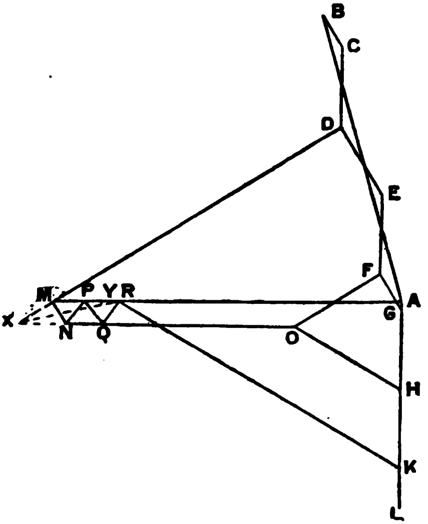


FIG. 31.—STRESS DIAGRAM FOR BOTH ROOF TRUSS AND SUBSTITUTED FRAME.

the given force. Then, where this bar cuts an outside member of the frame, apply a force having all its elements (excepting the point of application) the same as those of the given load.

The introduced bar is represented in Fig. 32*a* by the dotted line S T, and the applied force by the chain dotted line M A. The force M A will have the same action on the members of the frame through the bar S T, that the load at the joint O P Q T has, therefore the load must be left out after M A is applied.

Note.—The introduced bar might have been placed between the *joint where the load is acting and the rafter, and the equivalent force applied at the rafter end of the bar.**

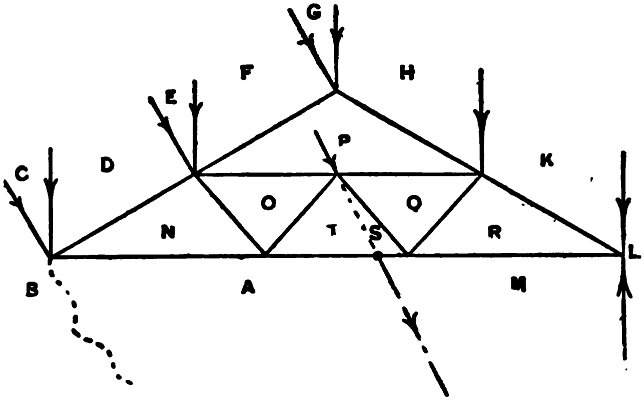


FIG. 32*a*.—FRAME DIAGRAM.
ROOF TRUSS, WITH LOAD AT INTERNAL JOINT.

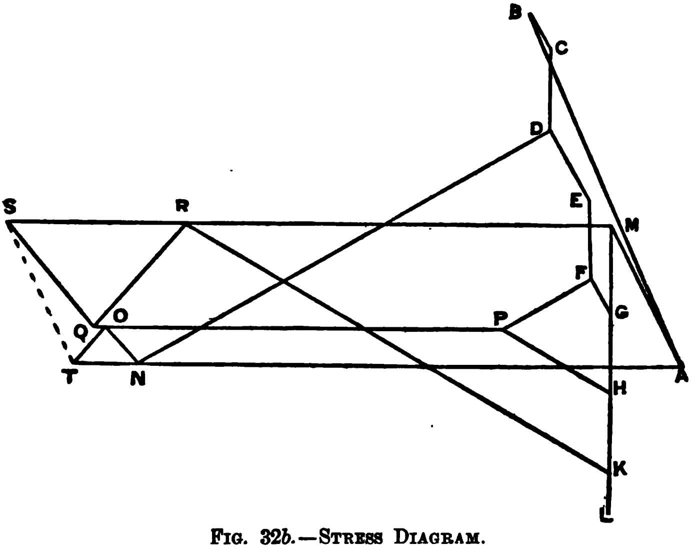


FIG. 32*b*.—STRESS DIAGRAM.

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* The student should work this method as an exercise. Some of the lines of the Stress Diagram will be lowered but the stresses will remain unchanged.

The Frame Diagram, Fig. 32*a*, is loaded similarly to the frame of Fig. 22. The reactions are therefore found in the same way.

After the external force polygon has been drawn, the Stress

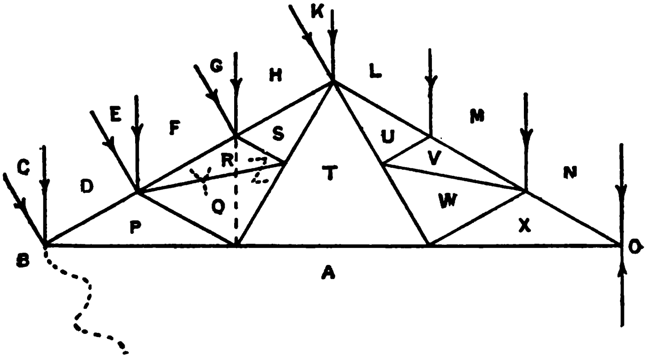


FIG. 33*a*.—FRAME DIAGRAM.

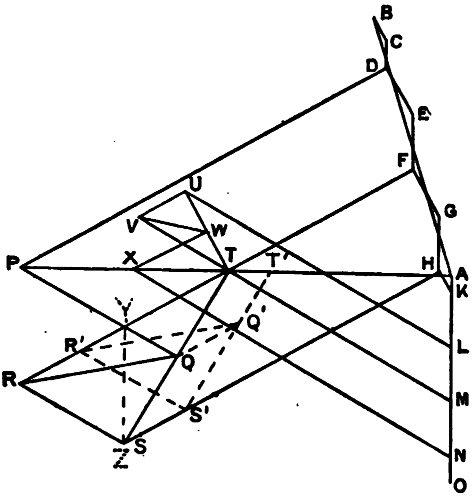


FIG. 33*b*.—STRESS DIAGRAM.

FRAME REQUIRING SPECIAL METHODS FOR SOLUTION.

Diagram may be completed by the same method as that used for the frame of Fig. 29.

Referring to the Stress Diagram of Fig. 32*a*, we see that the bar T S exerts the same pull at the joint T S M A as the force M A. These two therefore balance each other, and we are left with the pull which the bar S T exerts on the joint O P Q T. This pull S T, on the joint O P Q T, is identical in all its elements with the load; and therefore, the stresses in the members of the frame will be identical with those due to the original load.

Modified French Truss.—This Truss is illustrated in Fig. 33*a*, and presents some difficulties in its solution. We first determine the reactions as already explained and draw the external force polygon as in Fig. 33*b*. Secondly, we draw D P and A P parallel to the bars D P and A P respectively. This fixes the point P. But, although we know the point P, we can get neither Q nor R, nor any other point but X. This point X, however, does not help us, because we can proceed no further by aid thereof with the Stress Diagram.

FIRST METHOD OF OBTAINING THE STRESS DIAGRAM.—We know that the point R lies on a line drawn through F parallel to the bar F R and that S lies on a line drawn through H parallel to the bar H S. Now, assume a point R′ anywhere on the line F R and draw R′ S′ parallel to the bar R S and H S′ parallel to the bar H S. This fixes the point S′. Then S′ T′ and A T′ fix the point T′, and R′ Q′ and T′ Q′ fix the point Q′. Next move the figure R′ S′ Q′ parallel to itself keeping R′ on the line F R until Q′ lies on a line drawn through P parallel to the bar P Q. This is done by drawing Q′ Q parallel to F R so as to intersect P Q in Q. This determines the point Q. Then Q S and H S fix S and Q R and F R fix R and so on for the other points.

SECOND METHOD.—Substitute the bar Y Z (as shown by the dotted line in the Frame Diagram, Fig. 33*a*) for the two bars Q R and R S. This bar transfers the action of the loads at the joint G H S R F to the joint P Q T A; and therefore, the stress in T A will be unaffected. If the bar H S had been divided and similarly braced, then a bar from that joint to the joint P Q T A would enable a solution to be found.

In the Stress Diagram, Fig. 33*b*, we begin by finding the point P, then the points Y, Z, and T respectively. Having found the point T we can then proceed to find the other points in the same way in the previous cases.

THIRD METHOD.—First, find the stress in the bar T A, by taking one of the sections of the truss and thus obtain the resultant of the loads and the reaction of the wall on that section. Second, ascertain what stress in T A combined with the reaction of the other section of the truss on the apex will balance this resultant. Fig. 34 represents the right-hand section of the Truss of Fig. 33*a*. There is no wind pressure on this side. The loading consists of $\frac{1}{2}$ K L, L M, M N and N O in Fig. 33*a*. The resultant K′ O, Fig. 34, of these four loads passes through the centre of the rafter since N O is equal to $\frac{1}{2}$ K L and L M is equal to M N.

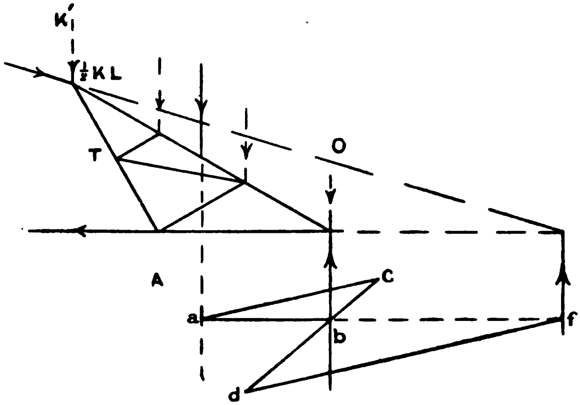


FIG. 34.—LOADS ON RIGHT-HAND SECTION.

We have now to find the resultant of K′ O and the reaction O A. Draw the line *a b* anywhere cutting the lines of action of the forces K′ O and O A as shown in Fig. 34. Through *b* draw *c d* at any angle to *a b*. Make *b c* represent to scale the force K′ O and *c d* to the same scale the reaction O A. Then join *a* with *c* and through *d* draw *d f* parallel to *a c*, so as to intersect *a b* produced in *f*. Then *f* is a point in the line of action of the resultant of the forces K′ O and O A. Its line of action is parallel to the lines of action of the forces K′ O and O A, and its magnitude is equal to their difference—that is, K′ A in Fig. 35.

Next produce the centre line of the bar T A in Fig. 34 to intersect the line of action of K′ A, the resultant of the two forces K′ O and O A. Then since the resultant K′ A, the action of the stress in the bar T A, and the action T K′ (*i.e.*, the reaction of the left-hand section on the apex of the frame) form a system of

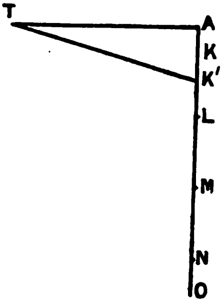


FIG. 35.—TRIANGLE OF FORCES ON SECTION.

forces in equilibrium. The line of action of the force T K′ must pass through the apex and the intersection of T A and K′ A, consequently an application of the triangle of forces will give the value of the stress in T A. This is shown in Fig. 35, where K′ is the centre of K L and the other points are points in the line of loads as in Fig. 33*b*. When this is known, the Stress Diagram can be completed.

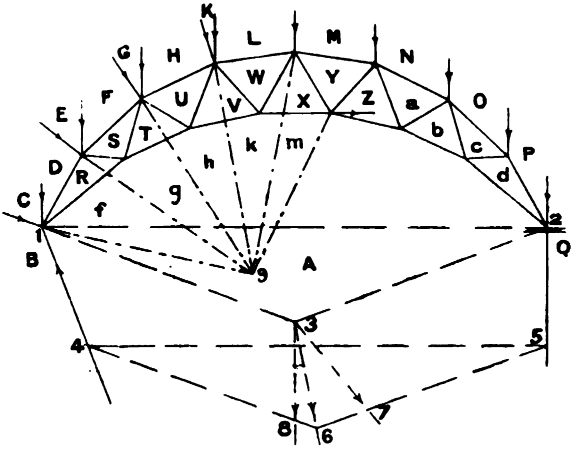


Fig. 36*a*.—Frame Diagram. Bowstring Truss.

Bowstring Truss.—There is no difficulty in drawing the Stress Diagram for this truss, but if we commence in the usual manner by drawing D R and A R, by the time we get to the other side, the finishing line would most probably not be parallel to its corresponding bar in the Frame Diagram. This is due to the short

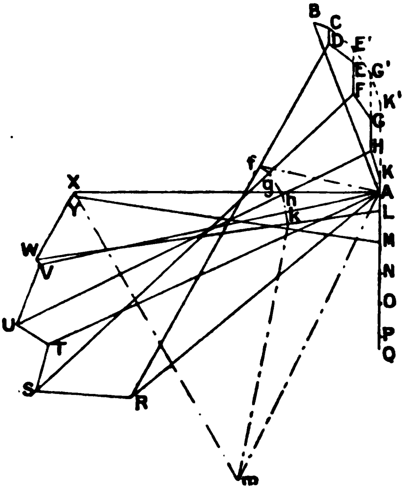


Fig. 36*b*.—Stress Diagram.

length of the bars R S, S T, &c., in the Frame Diagram and to the stresses in them being large compared to the loading on the roof as shown by the Stress Diagram.

We get a very much better diagram by determining the stress in one of the centre ties—*e.g.*, A X, by aid of a supplementary frame. Therefore, in order to find the reactions, the wind pressure may be supposed to act on a surface tangential to the curve of the roof at the joints; the length of the surface being equal to the sum of the two half bays on each side of the joint. The wind pressure at each joint will act along the radial line at the joint, and therefore the resultant of the wind pressures must pass through the centre of the outer curved flange.

The point 3 in the Frame Diagram, Fig. 36*a*, is the centre of the outer curved flange. This is a point in the line of action of the resultant wind pressure. This line is parallel to the line joining B with K′ in the Stress Diagram, and is represented by the line 3—7 in the Frame Diagram. B C, C E′, E′ G′, and G′ K′ represent the wind pressures B C, D E, F G and H K and therefore B K′ is the resultant in magnitude and is parallel to its line of action.

If the roofing is uniform, the centre of the curve of the outer flange will be a point in the line of action of the resultant load. Therefore, the resultant of the wind pressure and of the roofing weight will also pass through the point 3. The line of action of this resultant will be parallel to the line joining B with Q in the Stress Diagram—*i.e.*, along the line 3—6 in the Frame Diagram. The Truss is in equilibrium under the resultant load acting along the line 3—6. The line of action of the right-hand reaction is known, and the point of application of the left-hand reaction is also known. These three forces must pass through one point. Therefore the line joining the point 1 with the point where the line 3—6 cuts the line 2—5 will give the line of action of the left-hand reaction. But as the point of intersection of the lines 3—6 and 2—5 would be far off the paper we use the following construction:—Join the point 1 with 2, 2 with 3, and 3 with 1; then take any point 5 in the line of action of the right-hand reaction and draw 5—6 parallel to 2—3. Then draw 6—4 parallel to 3—1 and 5—4 parallel to 2—1 so as to intersect 6—4 in the point 4. If the point 1 be joined with the point 4 the the line 1—4 will pass through the point of intersection of the line 3—6 with the line 2—5. The line 1—4 is therefore the line of action of the left-hand reaction.

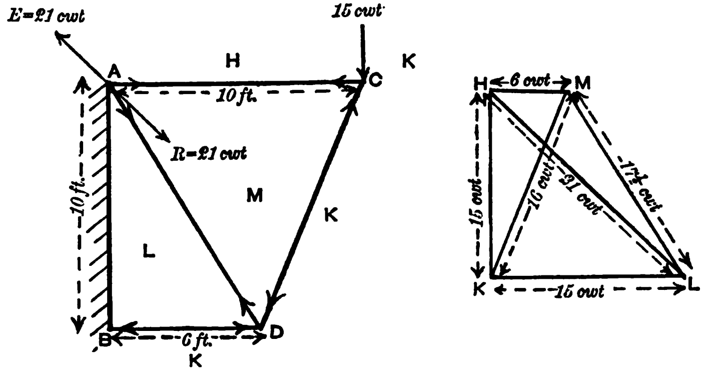
In the Stress Diagram, Fig. 36*b*, draw a line through the point B parallel to the line 1—4, so as to intersect the line Q A in the point A. This determines the reactions and enables us to proceed with the Stress Diagram.

We shall first determine the stress in X A. Assume a point 9 anywhere and connect it as indicated by the chain dotted lines in the Frame Diagram. This point 9, when connected with the joints of the outer flange, forms a supplementary frame in equilibrium under the following forces:—

(1) The wind pressure.
(2) The loads C D, E F, G H, K L and L M.
(3) The reaction A B.
(4) The action of the stress M Y on the joint L M Y X W.
(5) The action of the stresses in Y Z and Z A on the joint X Y Z A.

In the meantime, the internal bars R A, R S, S T, T A, on to W X and X A are left out of account.

In the Stress Diagram, Fig. 36*b*, we begin by drawing A*f* parallel to the supplementary bar A*f* and D*f* parallel to the bar D*f*. This fixes the point *f*. Then *f g* and F*g* give the point *g* and so on until the point *m* is obtained. Now draw *m* Y parallel to the bar *m* Y (which is coincident with the bar X Y) and M Y parallel to the bar M Y this fixes the point Y. In a similar way Y X and A X fix the point X, when the Stress Diagram may then be finished in the usual way. This method gives a more accurate estimate of the several stresses than by following the usual direct plan, as explained at the beginning of this example. Moreover, this construction is perfectly general in its application and may be used to determine the stress in any one bar of a frame.



To Illustrate Example I.

Example I.—A is a point in a wall 10 feet vertically over another point B. From A and B there project two horizontal bars, A C, B D (the former being 10 feet and the latter 6 feet long), and D is joined by two bars with A and C. If a weight of 15 cwts. be hung from C, find the stresses on all the bars, and show which are in tension. Find also the resultant stress on the point A. You may neglect the weights of the bars.

Answer.—In the figure we have denoted the spaces by letters according to Bow's notation. To obtain the Stress Diagram we must draw H K parallel to the force H K and 15 units long. Then make H M parallel to the bar H M and K M to the bar K M. This gives us the point M. M L parallel to the bar M L, and K L to the bar K L, fix the point L. H L, when joined, gives the reaction at the joint A, and the other lines the stresses in the bars. Their values are marked on the figure. A C and A D are in tension, while C D and B D are in compression.

LECTURE XXV.—QUESTIONS.

1. A king post truss, whose height is one-fourth of its span, is loaded at the joints with vertical loads of 15, 30, and 45 units respectively. Determine the nature and amount of the stresses in each member of the frame. (S. & A. Adv. Exam., 1896.)

2. A roof of 28 feet span, height 7 feet, rests on king-post trusses, 10 feet apart The weight of the roof is 20 lbs. per square foot. Find the stresses on each part.

3. If the above roof has a wind pressure of 40 lbs per square foot on one side, find the stresses on each part.

4. A roof of the form shown in Fig. 22, is 40 feet span and 10 feet high. The horizontal tie-bar is 8 feet below the vertex. Find the stresses in each part when loaded with 2 tons at each joint.

5. If, in the previous question, the maximum wind pressure on one side be 2 tons on each bay, find the stresses on all the bars.

6. Suppose both ends of the roof truss in Fig. 29 are anchored, and that in the substituted frame the bar X Y slopes in the opposite direction to that in Fig. 30; find the reactions and the stresses in the roof truss.

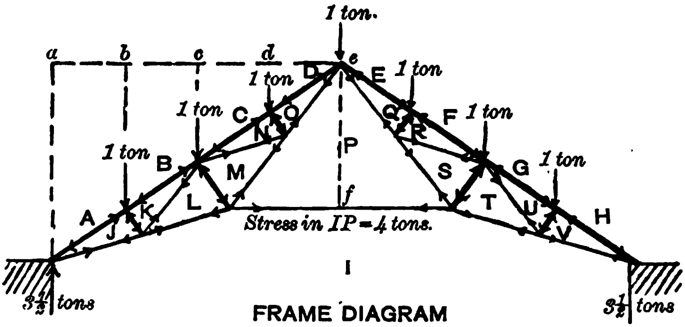
7. Work out the stresses for Fig. 32*a* by the method referred to in the second note.

8. Suppose both ends of the modified French struss in Fig. 33*a* are anchored, and that the substituted bar Y Z lies across the spaces V and W; find the reactions and stress (1) neglecting wind pressure, (2) when wind pressure is taken into account.

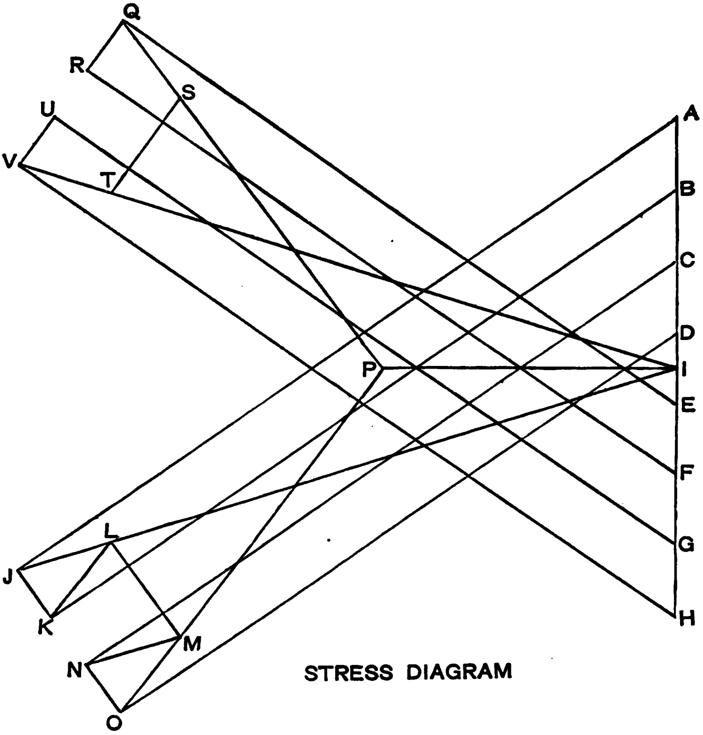
9. Find the stresses in the bowstring truss, shown in Fig. 36*a*, when both the ends are anchored (1) without wind pressure, (2) when wind pressure is taken into account.

10. The following figures give the Frame and Stress Diagrams for a French Truss. Verify the Stress Diagram and redraw it in the manner explained in the text.

11. Draw the Stress Diagram when there is a wind pressure of 4 tons on the left-hand slope, assuming both sides of the roof to be fixed to the walls.



FRAME DIAGRAM



STRESS DIAGRAM

ILLUSTRATIONS FOR QUESTIONS 10 AND 11.

LECTURE XXVI.

CONTENTS.—Deficient Frames—Iron King Post Truss—Queen Post Frame—Solution of the Second Method—Solution of the Third Method—Solution of the Fifth Method—Yieldingness—Questions.

Deficient Frames.—Iron King Post Truss.—In practice the foot of the king rod N O is made virtually solid, as shown by the full lines in Fig. 37*a*. If the short bars N A and O A are made freely jointed at N O A, it will be evident that the spaces N and O would change their shape if the loads at the centre of the rafters were unequal. This change of shape is resisted by making the joint N O A rigid.

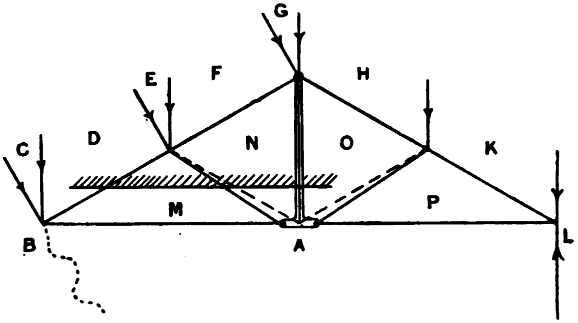


FIG. 37*a*.—FRAME DIAGRAM FOR DEFICIENT KING POST TRUSS.

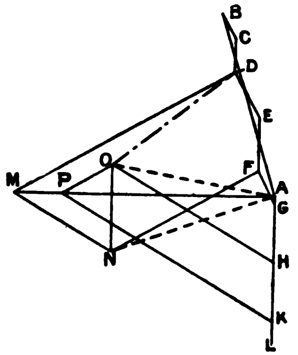


FIG. 37*b*.—STRESS DIAGRAM FOR DEFICIENT KING POST TRUSS.

Since the bars N A and O A are very short compared with the other members, we can draw the Stress Diagram as if the frame were made as shown by the dotted lines in Fig. 37*a*. The full lines in Fig. 37*b*, which is the Stress Diagram of Fig. 37*a*, are drawn on the above assumption.

If a line of section be drawn, beginning in one space of a Frame Diagram and ending in another, so as to pass through a joint or cross two or more bars, then the line joining the points in the Stress Diagram named after the beginning and end spaces gives the resultant of the stresses in all the bars meeting in or crossing that line.

The student can easily verify this by referring to the Stress Diagram.

In Fig. 37*a* a shaded line is shown beginning in the space D and ending in the space O. Then in Fig. 37*b* the chain dotted line D O is parallel to the line of action of the resultant of the stresses in the bars D M, M N, and N O. The length of the line D O gives the magnitude, and from D to O the way of the resultant with respect to the top side of this section. A point in the line of action of this resultant may be found by drawing a line through the joint D E F N M parallel to the line joining D with N in the stress diagram to cut the bar N O produced. The line D N is the resultant of the stresses D M and M N, which must pass through the joint D E F N M. The dotted line O A gives the resultant of the stress actions in the bars O P and P A on the imaginary joint N O A. Therefore, since the bar O A is very short, the force acting on the pin at the end of this bar will be approximately represented by the elements found from the line O A. Similarly, the dotted line N A gives the elements of the force acting at the end of the short bar N A.

The forces acting at the foot of the King Rod are represented by Fig. 38. These forces produce bending and tension in the parts O A, A N, and N O.

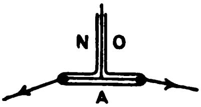


FIG. 38.—FOOT OF KING ROD.

Queen Post Frame.—A Queen Post Frame is represented in its normal position by the solid lines in Fig. 39. If the frame be

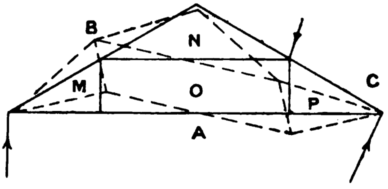


FIG. 39.—DISTORTION OF A FREELY JOINTED QUEEN POST FRAME.

freely jointed, it would be deformed into the shape represented by the dotted lines by a single force B C applied as shown at the joint N B C P O.

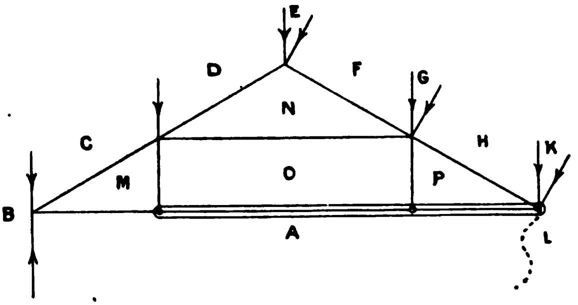


Fig. 40a.—Frame Diagram.

Queen Post Frame, with Part of Tie-Rod Made Continuous.

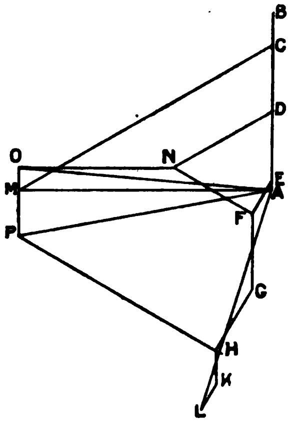


Fig. 40b.—Stress Diagram.

This change of shape may be resisted in several ways, such as the following :—

(1) By a diagonal in the central parallelogram. This diagonal would have to stand push if the wind caught the frame on one rafter, and pull if the wind pressure were on the other; or the stresses might be due to snow. It is the usual practice to put two diagonal ties in the parallelogram, so that when a push comes on one diagonal the other receives it as a pull. In drawing the Stress Diagram for such a frame, if a push comes on one of the ties, we omit that bar and take the other.

(2) By making the bar continuous between the joints M O A and P C A, and therefore able to resist being bent into the dotted form shown in Fig. 39.

(3) By making the whole tie-beam continuous. This causes the frame to become redundant; *i.e.*, it may be self stressed, by having the bars M O and O P of unequal length, or badly fitted.
(4) By making one rafter continuous.
(5) By making the rafters and tie-beam continuous. This is the usual form in actual practice and causes the frame to become redundant.

Solution of the Second Method.—The reactions are ascertained by a Substituted Frame as already explained. In the Stress Diagram, Fig. 40*b*, we begin by drawing D N and F N; C M and A M; N O and M O; O P and H P all parallel to their respective bars. Then P and O joined with A give the finishing lines of the Stress Diagram. The lines A P and A O are not parallel to the bars A P and A O. This indicates that there is bending in the continuous part of the tie-rod.

In Fig. 41, the forces are shown acting on the part of the tie-beam which is continuous. The vertical components of the forces A O and A P produce bending in the bar, while the horizontal components produce tension.

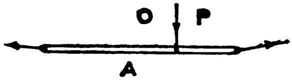


FIG. 41.—THE CONTINUOUS PART OF TIE-BEAM.

Solution of the Third Method.—Having found the reactions and drawn the External Force Polygon, as in Fig. 42*b*, we can then find the point N. We observe that O must lie on the line N O, which is drawn parallel to the bar N O; M must lie on the line D M drawn parallel to the bar D M, and P on the line K P drawn parallel to the bar K P.

On reference to Fig. 39 we see, that so long as the rafter ends always remain in the same horizontal line, the joint O P A must go down as much below the horizontal line as the joint M O A goes above it. Therefore, if the tie-beam is equally rigid along its length, the push required to distort it at the joint O P A must be equal to the pull distorting it at the joint M O A—that is, O P must be equal in length to M O in the Stress Diagram, Fig. 42*b*. If the tie-beam be unequally rigid, then the push and pull will be in proportion to the rigidity at the joints O P A and M O A in Fig. 39. In Fig. 42*a* the distorting force is on the left-hand rafter, and therefore the joint M O A will go down; consequently M O is subjected to push stress.

We can now proceed with the Stress Diagram in Fig. 42*b*.

Since M O is equal in length to O P, P and M must lie where the line D M intersects the line K P. Again, O P and M O are parallel to the bars O P and M O respectively, and N O is parallel to the bar N O. This fixes the point O. Joining the point P M and the point O with A we complete the stress diagram.

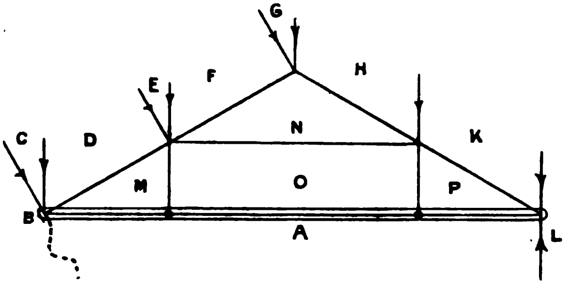


FIG. 42a.—FRAME DIAGRAM.

QUEEN POST FRAME, WITH CONTINUOUS TIE-BEAM.

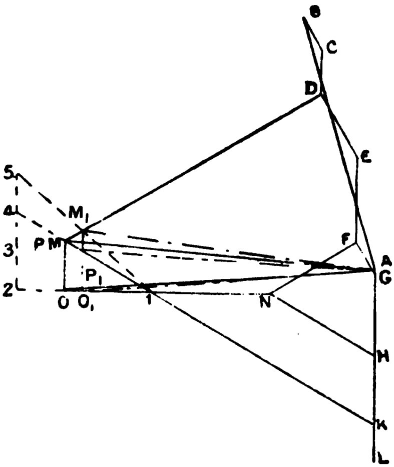


FIG. 42b.—STRESS DIAGRAM.

The forces acting on the tie-beam are illustrated by Fig. 43. The force O P and the vertical component of P A constitute a couple tending to produce clockwise rotation. The force M O and the vertical component of A M form another couple of equal moment, and also produce clockwise rotation. These two couples bend the beam, as indicated in Fig. 39. The horizontal components of the forces A M and P A produce tension in the tie-beam.

Now, suppose the rigidity of the tie-beam at the joint P O A to be $\frac{2}{3}$ of its rigidity at the joint M O A, then $O_1 P_1$ must equal $\frac{2}{3}$ of $O_1 M_1$. We must remember that the joint O P A is always as much above as M O A is below the horizontal line. A construction to determine M_1, P_1, and O_1 is shown by the dotted lines in Fig. 42*b*. The point 2 is taken anywhere in the line N O. The line 2—5 is drawn perpendicular to line N O, and the line K P is produced to cut the line 2—5 in the point 4. Then the length 2—4 must be to the length 2—5 as the rigidities at the joints. In other words, the line 2—5 is three when 2—4 is two, and therefore 4—5 is equal in length to half of 2—4. Now join point 5 with point 1. This line cuts the line D M in the point M_1, and by drawing M_1 O_1 parallel to the bar M O, and N O_1 parallel to the bar N O, we obtain the point O_1.

The finishing lines of the Stress Diagram, Fig. 42*b*, are obtained by joining M_1, P_1, and O_1 with the point A, and are shown by the chain dotted lines.

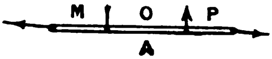


FIG. 43.—FORCES ACTING ON CONTINUOUS TIE-BEAM.

Solution of the Fifth Method.—In this arrangement of bars (Fig. 43*a*) if the joint F G H P O N descends through a small distance (say 1 inch) then the joint O P A of the tie-beam will descend 1 inch, the joint A M O will go up 1 inch and the joint M C D N O will rise 1 inch. Now, all this will take place irrespective of the rafters and tie-beam being of equal or of unequal yieldingness.

Yieldingness.—Two springs are of equal yieldingness, when they stretch through the same amount under equal loads.

One spring would have a yieldingness of three times another, if the first extended three times the amount that the second stretched under the same load.

Further, if two springs of equal yieldingness are attached to the same load, so that they each extend through the same amount; then each spring will carry one half of that load. But, if two springs of unequal yieldingness are attached to the same load, so that they each extend through the same amount; they will each carry a share of the load *inversely* proportional to their yieldingness. Suppose we have two springs, the first one stretches say 1 inch under a load of 3 lbs., while the second one extends 1 inch under 1 lb.; then, if these two springs are set to carry a load of 4 lbs., they will each extend 1 inch and the first spring will carry 3 out of the 4 lbs., while the second will carry the remaining 1 lb.

The above remarks apply equally to bars supporting a load between them, whether they are under a similar kind of stress or not. For example, suppose a beam is jointed to a rod attached to a rigid point above it; then, their yieldingness would be measured by the amounts they would each come down under the same load, as applied to each separately at the point where they are jointed to each other.

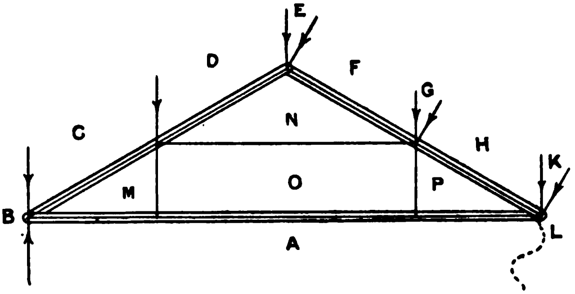


Fig. 43*a*.—Frame Diagram.

Queen Post Frame, with Rafters and Tie-Beam Continuous.

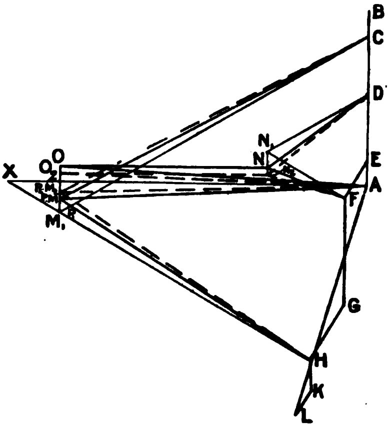


Fig. 43*b*.—Stress Diagram.

Referring to the Frame Diagram, Fig. 43*a*, we shall assume in the first place, that the yieldingness of the rafter at the centre in a vertical direction, is the same as the yieldingness of the tie-beam at the joint O P A, also in a vertical direction.* Therefore, whatever is the amount of the vertical component of the distorting force, they will each be subjected to the same stress.

Before we can do anything to the Stress Diagram, Fig. 43*b*, we must first find what amount of the distorting force passes into M O and O P in Fig. 43*a*, on the assumption that the rafters are freely jointed at their centres.

Figs. 44*a* and 44*b* show how this is done. In the Frame Diagram, the force Q G is the difference between the loads C D and F G in Fig. 43*a*, and G H is the same as in that figure. Now draw the Stress Diagram, Fig. 44*b*, in a similar manner to Fig. 42*b*. That is, Q G and G H are drawn to the same scale as the line of loads in Fig. 43*b*, when T will coincide with Q. Draw T U parallel to the bar T U, Q S to Q S and H V to H V. The pull S U is equal to the push U V; therefore S and V are at the intersection of Q S and H V. Then S U drawn parallel to the bar S U completes the Stress Diagram, Fig. 44*b*.

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* This does not mean that the rafter and the beam have equal rigidity.

The lengths of S U and U V, give the stresses in the Queen Rods S U and U V, on the assumption that the yieldingness of the rafters is infinitely large. But, the rafters have the same yieldingness as the tie-beam and therefore only half of the distorting forces will pass to the tie-beam. This means, that the pull in the Queen Rod M O and the push in the Queen Rod O P, are equal to one-half of S U and U V respectively.

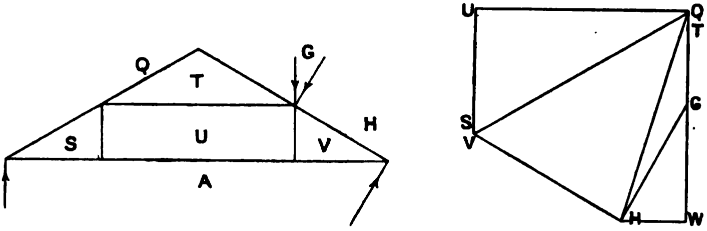


Fig. 44*a*.—Frame Diagram. Fig. 44*b*.—Stress Diagram.

Frame Carrying Distorting Forces only.

In the Stress Diagram, Fig. 43*b*, H P_1 and C M_1, are drawn parallel to H P and C M of Fig. 43*a* until they intersect. M_1O is drawn parallel to M O and A X is a horizontal line through A. On the line M_1O mark off two points O and M, where O is as much above A X as M is below it and the distance M O is equal to one-half of S U. This fixes the points M, O and P of the Stress Diagram, because P coincides with M.

Now, draw D N_1 and F N_1 parallel to the bars D N and F N until they intersect at N_1. Through N_1 draw N_1N parallel to M_1O. Then draw O N parallel to the bar O N, and we shall have found all the points in the Stress Diagram, Fig. 43*b*. On joining C with M; H with P; D with N; F with N; O with A; M with A; and P with A we finish the Stress Diagram. The dotted lines represent the Stress Diagram when the yieldingness of the rafters is less than that of the tie-beam.

Divide S U into two parts, having the ratio to each other that the yieldingness of the rafter bears to the yieldingness of the tie-beam. Then $O_2 M_2$ will have the smaller length as its value, if the yieldingness of the rafter is the smaller; and $O_2 M_2$ will have the larger length of S U if the yieldingness of the tie-beam is the smaller. For example, let the tie-beam be twice as yielding as the rafter. Then divide S U in the proportion of 2 to 1—*i.e.*, into three equal parts—and make $O_2 M_2$ equal to one of the three parts; keeping in mind, that O_2 is as much above A X as M_2 is below it.

LECTURE XXVI.—QUESTIONS.

1. Explain and prove the rule for obtaining the resultant of the stresses in all the bars of a frame crossing any given section. In Fig. 37*a* what is the resultant of the stresses in the bars A P, P K, and also in the bars A M, M N, and N F?

2. The dimensions of an iron king post truss for a roof are :—Span, 20 feet; height, 7 feet; distance between trusses, 8 feet. The roof weighs 12 lbs. per square foot. Find the stresses in each part.

3. In the above question find the stresses when the wind causes a pressure of 30 lbs. per square foot on one slope.

4. Explain the differences caused in the stresses by the different methods of completing a queen post frame. Mention some of the advantages and disadvantages of each.

5. A queen post roof has a span of 30 feet, and is 10 feet high. The roof weighs 10 lbs. per square foot, and the principals are 10 feet apart. Find the several stresses if the rafters and tie-beam are continuous.

6. If there is a wind pressure of 25 lbs. per square foot on the roof in Question 5, find the stresses in the bars.

LECTURE XXVII.

CONTENTS.—Wharf Crane—Example I.—Common Jib Crane—Balanced Jib Crane—Derrick or Scotch Crane—Foundry Crane—Sheer Legs—Example II.—130-Ton Steam Crane—Tables of Dimensions and Weights of 130-Ton Crane—Example III.—Questions.

Wharf Crane.—Suppose, as in Fig. 1*a*, that a single movable pulley carries the load W. Then, neglecting friction, the pull throughout the chain will be one half of W. Again, assume that the pull of the chain acts at the centre of the pulley or barrel round which it may be passing. In the Frame Diagram, Fig. 1*a*, the external forces acting on the frame are all duly indicated. At the Jib end, there are two forces—viz., the pull of gravity D E, on the supported mass, acting vertically downwards and the force C D, due to the pull in the chain which is assumed to be parallel to the tie-rod C H. At the top end of the vertical post, there are the forces B C and A B acting as shown, which are both due to the pull in the chain on the pulley at the post head. There is also a force acting at the centre of the barrel along the crane post. This force must be transferred to the footstep as shown and is called E F in the diagram. There is also a pressure transmitted by the sole plate to the vertical post. This pressure is

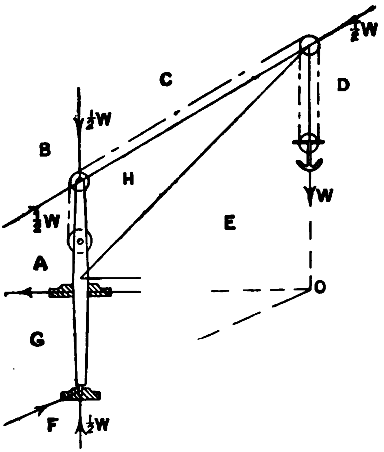


FIG. 1*a*.—FRAME DIAGRAM.

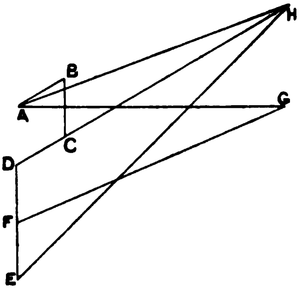


FIG. 1*b*.—STRESS DIAGRAM.

WHARF CRANE.

assumed to have its line of action horizontal and is lettered G A. Finally, we have the action of the footstep on the lower end of the vertical post.

The three forces D E, F G and G A, must form a system in equilibrium. Therefore, since the lines of action of D E and G A are known, if we produce them to meet in O, then the line of action of F G is known because it must also pass through O.

The Stress Diagram, Fig. 1*b*, may now be drawn. Draw the line of loads A B, B C, C D, D E, and E F. Now, draw F G parallel to the line of action of the force F G and A G parallel to the line of action of the force A G. These close the External Force Polygon. Then if C H and E H be drawn parallel to the bars C H and E H respectively they fix the point H. The line joining the point A with the point H is the finishing line of the Stress Diagram. This line is not parallel to the bar A H because the bar A H is subject to bending.

In Fig. 2, we have a representation of the forces acting on the vertical post; from which, we can determine the bending, tension, and compression stresses in the crane post.

The lengths of the lines C H and E H in the Stress Diagram, Fig. 1*b*, give the stresses in the tie-rod and jib respectively. The horizontal component of G F gives the shear on the bolts of the footstep and G A the shear on the bolts of the sole plate.

EXAMPLE I.—In a wharf crane the post, tie-rod, and jib measure 15, 20, and 30 feet respectively, what would be the nature and amount of the stresses in each of the three members when a load of 7 tons is suspended over the pulley at the jib head, (1) when the lifting chain passes from the pulley to the drum or barrel parallel with the jib, (2) when the drum is placed so that the chain passes from the jib head parallel with the tie-rod? (S. and A. Exam., 1890.)

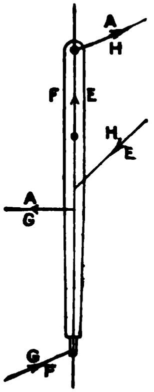
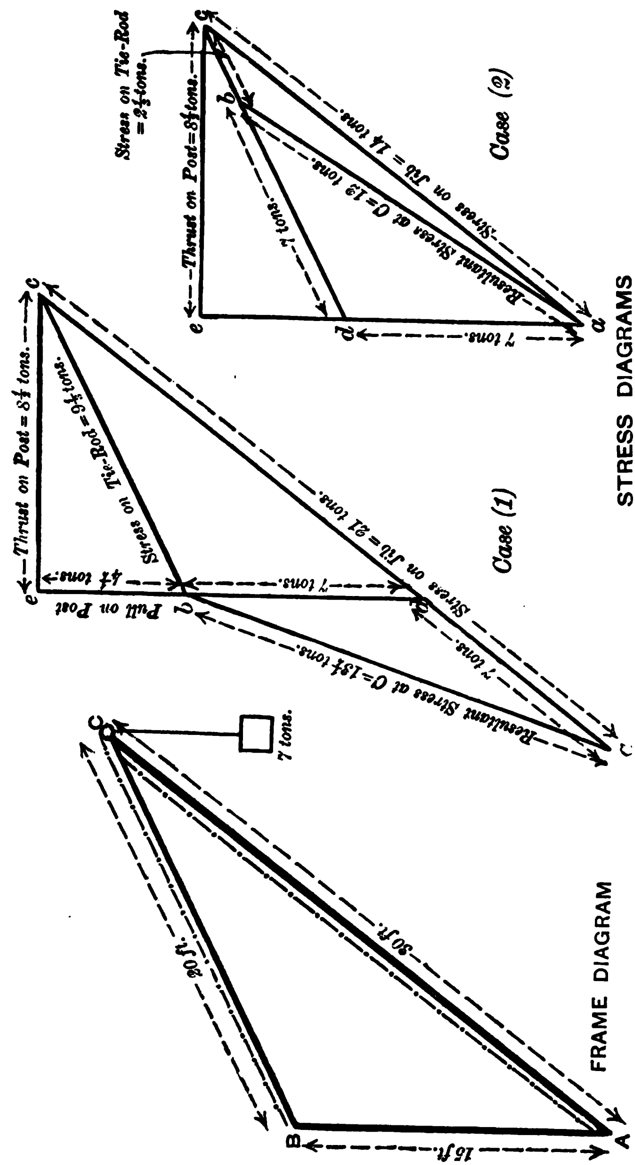


FIG. 2.—FORCES ACTING ON VERTICAL POST OF CRANE.

ANSWER.—First, draw to scale a Frame Diagram A B C, as shown. This will be coincident with the centre lines of the different members of the crane.

Case (1).—Here the lifting chain passes from the pulley at the jib head parallel to the jib, and, neglecting the friction of the pulley, we shall have two equal external forces at the joint C due to the tension in the two parts of the chain.

In order to draw the Stress Diagram, we may first proceed to



STRESSES IN A JIB CRANE.

determine the resultant of these two forces and consider it as a single external force applied to the joint C, and then draw the triangle of forces. Or we may at once draw the polygon of forces for the joint. Thus, draw *b d* to represent the load of 7 tons, and *d a* equal to *b d*, and parallel to C A, to represent the tension in that part of the chain over the pulley; then drawing *a c* and *b c* respectively parallel to A C and B C, we determine the point *c*, and therefore the magnitude of the stresses in the jib and tie-rods. These will evidently be compression and tension respectively. If we join *b a* we obtain the resultant external force at the jib head, and *b a c* will be the triangle of forces determining the same stresses as above.

The nature and amount of the stress in the post will depend on the mode of fixing it. It is evident that the pull *b c* in the tie-rods may be resolved into a vertical component *b e*, producing tension in the post, while the horizontal component *e c* represents the force tending to bend the post round A.

Case (2).—Here the chain passes from the pulley parallel to the tie-rods. We proceed as before, and draw *b d* to represent the pull in the part of the chain above the pulley, and *d a* the pull in the vertical part of it, then *a c* and *b c* drawn parallel to A C and B C respectively, determine the point *c*, and represent the stresses in the jib and tie-rod. Again joining *b a*, we see that this represents the resultant external force at the pulley. The remainder of the diagram is the same as in Case (1).

The magnitudes of the different stresses are shown on the diagrams, and enable us to compare the relative merits of the two arrangements. Thus, if we suppose the load to be suspended from the end of the jib without the intervention of a pulley, we get *b d c* in Case (1), or *d a c* in Case (2) as the corresponding Stress Diagram. The effect of introducing the chain and pulley is in Case (1) to *increase* the thrust in the jib by 7 tons—*i.e.*, the pull in the chain—without affecting the pull in the tie-rods, while in Case (2) the effect is to *diminish* the pull in the tie-rods by the same amount—7 tons—without increasing the thrust in the jib. Thus, other things being equal, Case (2) is the better arrangement.

Common Jib Crane.—In the common Jib Crane represented in Fig. 3, the movable pulley has one sheave, and the chain passes direct to the barrel from the Jib-head. The barrel is carried by the cast-iron framing. There are two tie-rods inclined at an angle θ degrees to the centre line of the crane as shown by the plan of the tie-rods. We assume the cast-iron frame to be freely jointed where the tie-rods and the jib meet it, and also where the horizontal part meets the upright post.

The forces acting on the frame of the crane are indicated in Fig. 4*a*. At the Jib-head there are the two forces W and ½ W. At the point in the bar G H, representing the centre of the barrel, there is a force ½ W, indicated by the dotted line and arrow head. The lines of action of the dotted force and the force B C are coincident. They lie along the line joining the centre of the Jib-head pulley and the chain barrel.

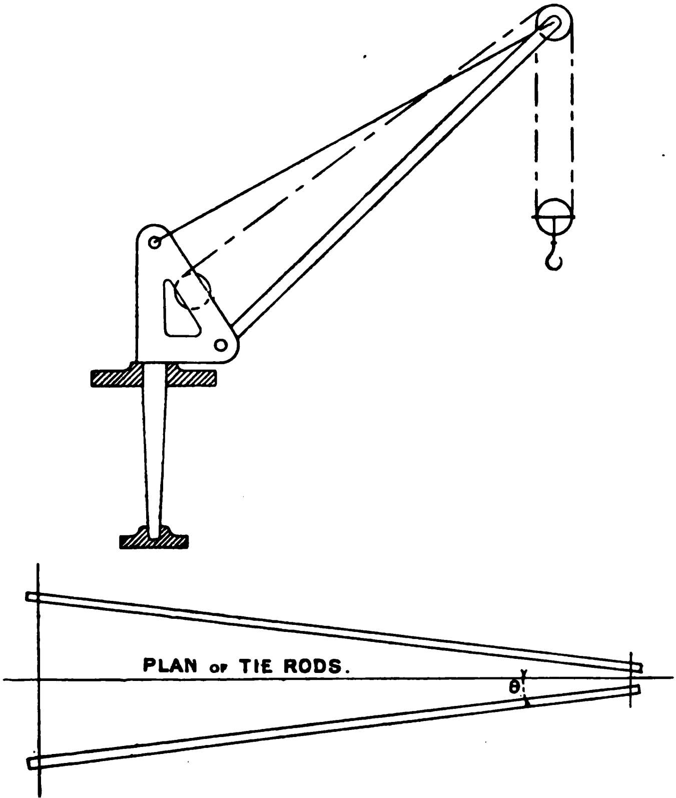


Fig. 3.*—Outline Diagram of Common Jib Crane.

We have here an example of a force acting at a point in a bar. The force acting at a point in the bar G H as represented by the chain dotted line, is replaced by the two equivalent parallel forces D E and A B applied as shown. Their magnitudes will be inversely as the lengths into which the bar G H is divided while the sum of their magnitudes is $\frac{1}{2}$ W.

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* The Plan of the Tie-Rods of this crane has been drawn to a larger scale than the crane itself.

Before beginning the Stress Diagram, we must first determine the values of A B and D E. Lay down a line to measure $\frac{1}{2}$ W and divide this line in the same proportion as the bar G H is divided by the line of action of the dotted force. Then, these two parts will measure to the same scale as the whole line, the respective values of the forces A B and D E. The greater force is placed at the end of the shorter division of G H.

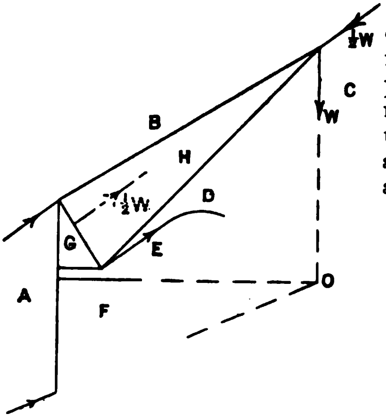

FIG. 4a.—FRAME DIAGRAM.

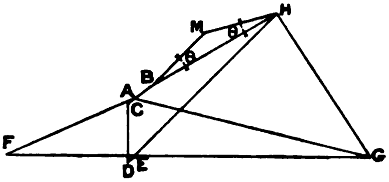

FIG. 4b.—STRESS DIAGRAM.

COMMON JIB CRANE.

As in the last crane, the pull of gravity C D on the supported mass, the pressure of the sole-plate E F on the upright and the reaction of the footstep F A on the upright, are in equilibrium and therefore their lines of action all pass through one point O. The line of action of E F is assumed to be horizontal and the line of action of C D to be vertical. In the Stress Diagram, Fig. 4*b*, we first draw A B, then B C, C D, and D E. This completes the line of loads. Now, from E draw E F parallel to the line of action of the pressure E F; and, from A draw A F parallel to the line of action of the reaction A F. These two lines determine the point F and complete the External Force Polygon. Then, draw B H and D H parallel to the tie-rod B H and the jib D H respectively. These fix the point H. Draw H G and E G parallel to the bars H G and E G. This fixes the point G. On joining C with G the Stress Diagram is completed.

B H in Fig. 4*b* represents the stress on the tie-rod, on the assumption that there is only one tie-rod lying along the centre line of the crane. We require, therefore, to resolve this stress into two components, one along each tie-rod. This is done in the Stress Diagram by drawing from H and from B lines H M and B M each making an angle with B H of θ degrees. Then the lengths of H M and B M measure to scale the stresses in the tie-rods.

Balanced Jib Crane.—The balance weight B W acting at G, is usually mounted on rollers in order that it may be moved nearer to the central post A when the load W is reduced. In this way the moments of the load and balance weight may be kept in equilibrium and thus prevent any undue bending action on the post at A. We may here remark that the balance weight B W at G and the load W at C are not necessarily equal.

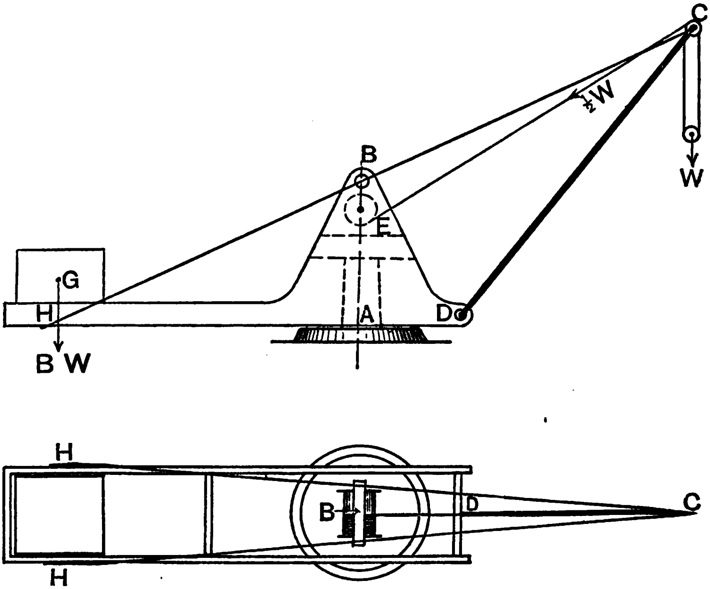

Fig. 5.—Balanced Jib Crane.

A single movable pulley carries the load W and therefore the tension throughout the chain is $\frac{1}{2}$W. We assume that bars join B with D; D with A; A with B; and A with H. These are all indicated in the Frame Diagram of Fig. 5. The line joining the centre of the pulley C with the centre of the chain barrel E is considered as the line of action of the stress in the chain. From the plan it will be seen that there are two tie-rods inclined to each other; these rods are often made continuous from C to H.

The load of $\frac{1}{2}$ W acting on the bar F G, Fig. 6*a*, is divided as explained in the previous example into forces B C and K A. If W is known, we can find on completing the Stress Diagram, the magnitude of the balance weight A B required to balance the moment of the load W about the joint F G E K A. Or, if the balance weight be moved nearer the crane post we can find what weight may be placed in the crane hook.

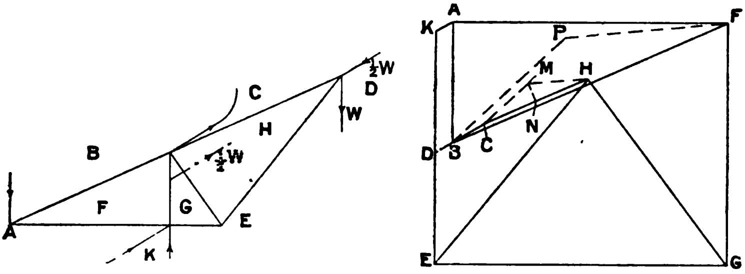

FIG. 6*a*.—FRAME DIAGRAM. FIG. 6*b*.—STRESS DIAGRAM.

BALANCED JIB CRANE.

We begin the Stress Diagram, Fig 6*b*, with the line B C, then C D, and D E. We next find the point H, then, G, F, A, and finally K.

The actual stress in the tie-rods is found, by drawing H M and C M at an angle with H C, equal to half of the real angle between the tie-rods, as previously explained. From F and B draw F P and B P inclined to F B at half the angle between the parts of the tie-rods carrying the balance weight. If the tie-rods are continuous, then H M and F P are parallel.

Derrick or Scotch Crane.*—In Fig. 7, A B is the central upright post, capable of turning round A and B. B C is the jib and A C the tie-rod, which is usually a chain for raising or lowering the jib. The vertical post A B is kept upright by the back stays A E and A E_1. These stays are sometimes anchored to the ground, but are generally attached to the bars B E and B E_1. Boards are placed over these bars and stones or pig iron are placed thereon to act as counterweight to the load W. This crane is similar to that in Fig. 5, in

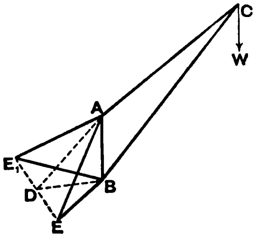

Fig. 7.—DERRICK CRANE.

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* See figure at end of this Lecture.

that back balance weights are used. The weights in Fig. 7 are, however, not required to be made movable in order to produce a moment round B equal and opposite to the moment of W round the same point. This is due to the fact, that the under sides of the bars B E and B E_1 rest upon the ground, and, therefore, no matter how much the moments of the balance weights round B may exceed the moment of the load, there is no bending stress produced on the pivot at B.

The chain carrying the load W is usually parallel to the tie-rod A C. It then passes round a pulley at A and down to the barrel on the upright post.

By reference to the Stress Diagrams, Figs. 1*b* and 6*b*, the Stress Diagram for Fig. 7 may be drawn. When the plane of the triangle A B C in Fig. 7, coincides with the plane of the triangle A B E, the stress in A E will be a maximum, the stress in A E_1 will be theoretically zero, and the weight required at E may then be found. Similar considerations will give the stress in A E_1 and the weight required at E_1.

Let the plane of the triangle A B C now occupy any intermediate position between the planes of the triangles A B E and A B E_1. Then the stresses in A E and A E_1, may be found by producing the plane of the triangle A B C to intersect the planes A E_1 E and E_1 B E in A D and B D. Now proceed to find the stress in A D as if D were anchored to the ground, then resolve this stress along the stays A E and A E_1, as explained for the inclined tie-rods of the two previous examples. The angle E_1 B E is usually a right angle.

Foundry Crane.—The Frame Diagram, Fig. 8*a*, illustrates the arrangement of the parts of this type of crane. The pulley carrying the load W is attached to a small bogie running between two parallel horizontal beams. The external forces acting on the crane are F G, G A, A B, C D, E F (each equal to W), B C, and D E. The external force A B balances E F, and C D balances G A, therefore the remaining external forces B C, D E, and F G form a system in equilibrium. Their lines of action will pass through one point O, and their magnitudes may be determined by the triangle of forces. These results may be made use of in drawing the Stress Diagram, Fig. 8*b*. Commence by drawing E F, F G, G A, and A B all equal in magnitude to W, and parallel to their respective lines of action. If we draw B C parallel to the line of action of the force B C, so as to intersect the line through E parallel to the line of action of the forces C D and D E in the point C, then by marking off C D equal to W we complete the external force polygon.

The stress in A H will be unaffected by raising or lowering either the footstep or the bearing at the top, so long as the line of action of the load and the position of the joints 1, 2, and 3 remain unchanged. Let us suppose that the footstep coincides with the point 1, and the bearing at the top with the point 3, then the line of action of the stress in C H will be along the centre line of the bar C H, while the line of action of the footstep reaction B C_1 will pass through the point 1 and the centre of the pulley carrying the load.

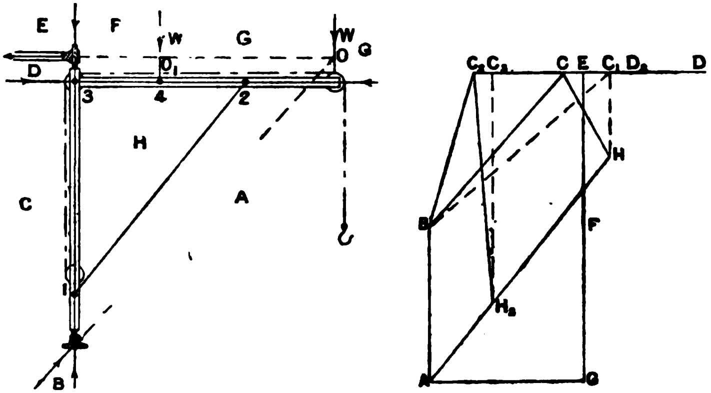


FIG. 8a.—FRAME DIAGRAM. FIG. 8b.—STRESS DIAGRAM.

FOUNDRY CRANE.

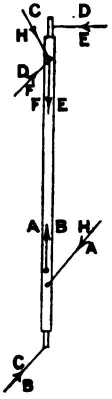


FIG. 9.—FORCES ACTING ON UPRIGHT OF FOUNDRY CRANE.

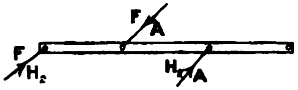


FIG. 10.—FORCES ACTING ON JIB WITH PULLEY AT POINT 4 ON FIG. 8a.

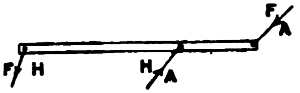


FIG. 11.—FORCES ACTING ON JIB IN FIG. 8a WITH PULLEY AT END OF SAME.

In the Stress Diagram, Fig. 8*b*, draw B C_1 through the point B parallel to the line joining the point 1 with the centre of the pulley carrying the load, so as to cut the line C D in the point C_1; then by drawing C_1 H and A H parallel to the bars C H and A H respectively, we determine the point H. Finally, C joined with H finishes the Stress Diagram. When the pulley carrying the load occupies the position represented by the point 4, then C_2, D_2, and H_2 (found in a similar manner) are the corresponding points in the Stress Diagram.

Fig. 9 represents the forces acting on the upright, which produce bending, tension, and compressive stresses, for the case when the movable pulley is at the end of the horizontal jib.

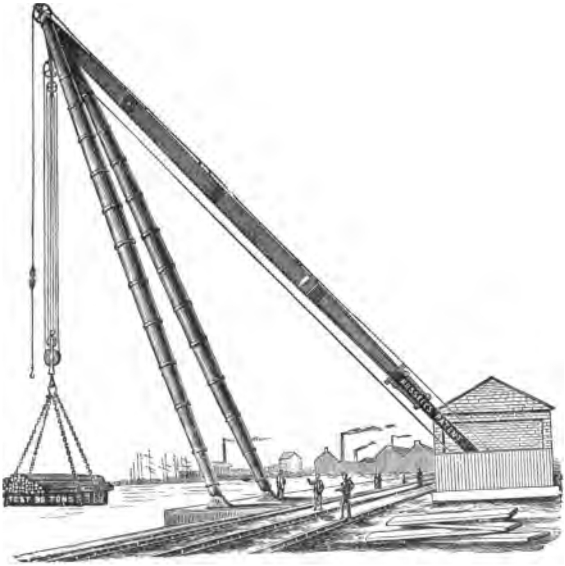


FIG. 12.—RUSSELL'S PATENT SHEER LEGS.

In Figs. 10 and 11, if the maximum bending moment in each case be the same, then the point 2 in Fig. 8*a* has been so chosen, as to make the bending moment on the horizontal jib have its least value whilst the pulley carrying the load passes from one end of the jib to the other. (See Ques. 41 of Hons. S. & A. Exam. in Machine Constn., 1890.)

Sheer Legs.—A common appliance for lifting engines and boilers into ships is the sheer legs or sheers. The illustration shows one that has been erected at West Hartlepool by Messrs. George Russell & Co., of Motherwell, for a load of 80 tons overhung 38 feet 6 inches. It consists of two tubular front legs, each 105 feet long, swinging upon pins at their lower ends, and connected together at the top, which is supported by a hollow stay or back leg. This stay is fixed to the gunmetal nut of a forged steel screw, which rotates inside the back leg. The screw is anchored at its lower end, and can be rotated by a hydraulic engine. As the screw revolves one way or the other, the back leg is shortened or lengthened, and the top is moved in or out, as shown on Fig. 13. The total horizontal travel thus given to the load is 50 feet.

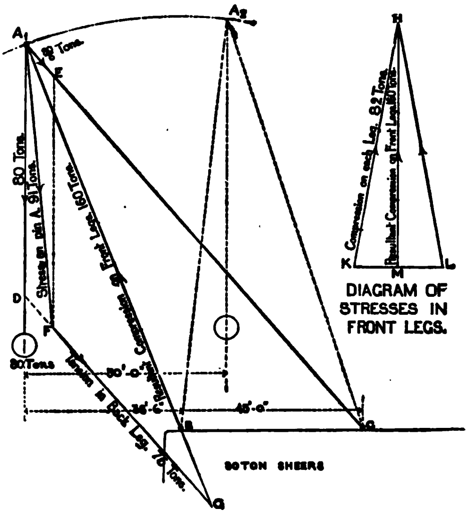
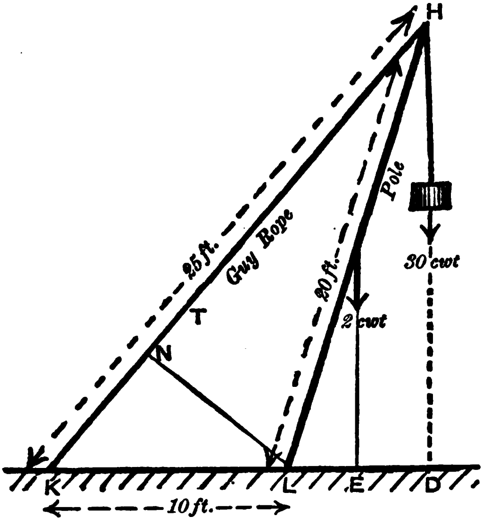


FIG. 13.—COMBINATION OF FRAME AND STRESS DIAGRAMS FOR FIG. 12.

Chains worked by a pair of hydraulic engines are used for lifting, and there are separate chains for light and heavy loads. The latter chain operates a six-purchase pulley block.

In Fig. 13, A_1 B is the line of the front legs hinged at B, and A_1 C that of the back leg. The top can move between A_1 and A_2 by altering the length of A C. The vertical A_1 D represents the load on the sheers (80 tons), and A_1 E the tension in the chain, ($\frac{1}{6}$ of 80 or $13\frac{1}{3}$-tons since there are six chains supporting the load). Draw E F parallel to A_1 D, and D F parallel to A_1 E. Then A_1 F is the resultant force to be balanced by the stresses in the legs A_1 B, A_1 C. Draw F G parallel to A_1 C, and meeting A_1 B produced, if necessary, in G. Then A_1 G (160 tons) is the compression transmitted to the front legs, and F G (76 tons) the tension in the back leg. As the two front legs are not parallel, we must, in order to determine the actual stress in each, draw a second figure, as shown at the right-hand side. Here H M is equal to half A_1 G, and K H M and L H M are each half the angle between the front legs. K L being drawn perpendicular to H M, we obtain H K and H L (82 tons) as the compression in each leg.

Example II.—The foot of a uniform derrick pole of weight 2 cwts. rests on the ground, and the pole carries a weight of 1½ tons suspended from its upper extremity. The length of the pole is 20 feet, and it is kept in position by a guy rope fastened to the ground 10 feet to the rear of the foot of the pole and 25 feet in length. Find by construction or otherwise the tension of the guy rope.



Derrick Pole.

Answer. — In the figure, L H is the derrick and K H the guy rope. The weight of the derrick acts at its centre, and may be replaced by 1 cwt. at each end, so that the total load in the upper end may be taken as 31 cwts. This is the force A B on the Frame Diagram. For the Stress Diagram draw A B parallel to the force A B and equal to 31 units. Make B C parallel to the pole and A C to the guy rope. Then, A C is the tension in the guy rope and is equal to 25 cwts. and B C (= 53 cwts.), the compression in the pole.

This problem could have been solved by the Principle of Moments, as follows:—

Let T be the Tension in the guy rope.
„ w „ Weight of the pole.
„ W „ Weight hung from pole.

Drawing L N perpendicular to K H and taking moments about L (in the first figure), we have:—

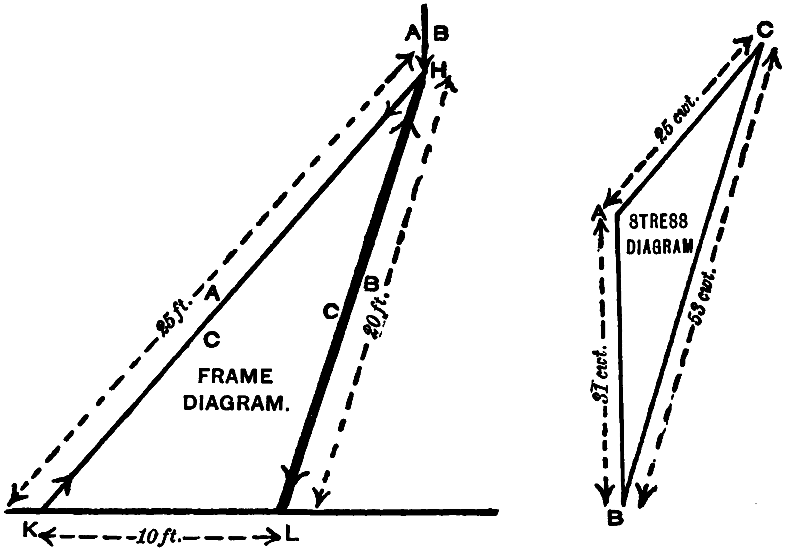
$$\mathrm{T \times LN = W \times LD} + w \times \mathrm{LE}$$

Or, $$\mathrm{T = W \times \frac{LD}{LN}} + w \times \mathrm{\frac{LE}{LN}} = (\mathrm{W} + \tfrac{1}{2}w)\,\mathrm{\frac{LD}{LN}}.$$

From Euclid II., 12, we have :—

$$\mathrm{KH^2 = KL^2 + LH^2 + 2\,KL\,.\,LD}$$

$$\therefore\quad \mathrm{LD} = \frac{\mathrm{KH^2 - KL^2 - LH^2}}{\mathrm{2\,KL}} = \frac{625 - 100 - 400}{20} = 6{\cdot}25 \text{ ft.}$$



Stresses in a Derrick Pole.

Also, $\mathrm{LN \times KH}$ = twice triangle $\mathrm{LKH = HD \times KL}$

Or, $$\mathrm{LN \times KH} = (\sqrt{\mathrm{LH^2 - LD^2}}) \times \mathrm{KL}$$

$$\therefore\quad \mathrm{LN} = \frac{\mathrm{KL}}{\mathrm{KH}}\sqrt{\mathrm{LH^2 - LD^2}} = \frac{10}{25}\sqrt{400 - 39}$$

Or, $$\mathrm{LN} = \frac{2}{5}\sqrt{361} = \frac{38}{5} = 7{\cdot}6 \text{ ft.}$$

Hence, $$\mathrm{T} = (30 + 1)\,\frac{6{\cdot}25}{7{\cdot}6} = \frac{31 \times 6{\cdot}25}{7{\cdot}6} = 25{\cdot}5 \text{ cwts.}$$

130-Ton Steam Crane* (*see Frontispiece*).—As an example of a very large crane, we have illustrated in the frontispiece to this volume the 130-ton steam crane erected for the Clyde Trustees at Finnieston Quay, Glasgow, by Messrs. Cowans, Sheldon, & Co., Limited, of Carlisle. A similar crane has also been put up at the new Cessnock Dock, Glasgow. The jib of this crane is made up of two steel tubular girders braced together by diagonal and cross stays. The tension rods have been sawn out of solid steel plates, and were not heated during their manufacture. They are connected to the jib by stays at intervals along their length. The foot of the jib is attached to one of the bottom corners of a large vertical triangular frame, and the tension rods to the upper corner, while the back one supports the balance weight which is placed between the two sides of the main framing. The boilers and engines are also placed within this framing, which is covered in so as to form an engine-house. The whole is fixed on the top of a circular base, which can rotate around a large central pin, and rests on steel rollers running on a steel pathway on the top of the foundation. There is also a roller bearing between the base and top of the centre-pin. The foundation, which is square in plan, is of concrete with granite corners and cope, and is supported on twenty-two concrete cylinders sunk into the sand. The centre piece of the crane is fixed to the foundation by six steel bolts cottered to washer plates in a tunnel inside the foundation.

There are two separate lifting blocks, the one for heavy, and the other for light weights. Each of these can be raised or lowered at two different speeds. Separate engines are provided for each of these blocks, and for rotating the crane. All three sets of engines have two cylinders with cranks at right angles, so as to start from any position. Steel wire ropes are used for hoisting instead of chains. The heavy weights are taken on an eight-purchase pulley block, and the light weights on a double-purchase pulley block. All the gearing, up to 24 inches diameter, is of cast steel, and the remainder of a mixture of cast-iron and steel. Gun-metal bushes are used throughout.

The crane is provided with a 160-ton Duckham hydrostatic weighing machine, and was tested by loading it with 150 tons of steel rails. Its radius of action is 65 feet, and the total lift is 100 feet.

The following tables show some of the leading dimensions and weights:—

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* For a complete description of this crane see *Engineering*, June 9, 1893.

PARTICULARS OF 130-TON CRANE AT FINNIESTON QUAY, GLASGOW.

Part.	Number.	Length.	Breadth.	Diameter.	Thickness or Height.	Height above or below Quay.	Total Weight in Tons.	Projects beyond Cope of Quay Wall.
Foundation above front cylinders,	...	40′ 0″	40′ 0″	...	45′ 0″	Top 20′ up	4,300	5′ 6″
Foundation, Concrete cylinders,	22	36′ 6″	...	9′ 7⅛″ and 5′ 9½″	...	...	...	...
Tunnel and Passage,	...	...	2′ 0″	Centre 10′ 9″	6′ 0″	Top 10′ down.	...	...
Foundation bolts,	6	38′ 9″	...	0′ 5″	...	...	8	...
Washer Plates,	6	6′ 0″	6′ 0″	. .	...	10′ down.	13	...
Centre Piece for Crane,	...	...	...	...	...	...	9	...
Centre Pin,	...	...	...	0′ 17″	...	...	6	...
Roller Path,	...	...	...	33′ 0″	...	...	12	...
Rollers,	75	...	...	0′ 14″	...	...	10·5	...
Framing,	...	...	...	...	27′ 0″	...	50	...
Boiler,	...	...	...	6′ 0″	14′ 0″	...	6	...
Back Balance,	...	...	...	...	...	...	100	...
Jib,	...	90′ 0″	...	Tubes 3′ 3″ at centre to 2′ 6″ at ends.	$\frac{7}{16}$″ at centre to ½″ at ends.	Top 110′ up.	45	...
Single Tension Rods,	...	...	0′ 10″	...	0′ 2¾″	...		
Double ,,	...	...	0′ 10″	...	0′ 1⅛″	...	15	...
Pins for ,,	...	...	...	0′ 8″	...	...		
Pulley for light weights,	...	...	...	2′ 6″	...	107′ 6″	...	44′ 3″
Pulley for heavy weights,	...	...	...	5′ 3″	...	100′ 0″	...	39′ 6″
Gin block for light weights,	...	...	...	...	...	...	...	44′ 3″
Gin block for heavy weights,	4 pulleys	12′ 0″	7′ 0″	Pulleys 5′ 3″	3′ 0″	...	7	39′ 6″
Hoisting Drum for light weights,	...	...	...	2′ 6″	...	...	...	...
Hoisting Drum for heavy weights,	...	10′ 0″	...	5′ 2″	...	...	10·5	...
Gearing,	...	...	...	...	...	...	8	...
All Castings,	...	...	...	...	...	...	120	...
Crane in working order, exclusive of back balance,	...	...	...	...	...	...	270	...
Radius of sweep for light loads,	...	69′ 9″	...	...	...	...	...	...
Radius of sweep for heavy loads,	...	65′ 0″	...	...	...	...	...	...

PARTICULARS OF ENGINES FOR 130-TON CRANE.

For Load.	Engines.			Distance Raised in One Minute.	Time to make Complete Revolution of Crane.
	No. of Cylinders.	Diameter.	Stroke.		
130 tons, . .	2	12"	16"	4′	5 mins.
60 ,, . .				8′ 10½"	2 mins. 17 secs.
20 ,, . .	2	8"	12"	28′	...
8 ,, . .				60′	...
Revolving, . .	2	8"	12"	...	...

PARTICULARS OF STEEL-WIRE ROPES FOR 130-TON CRANE.

Rope.	Inner Core.		Second Layer.		Third Layer.		Outer Layer.		Total No. of Wires in each Strand.	No. of Strands.	Total No. of Wires.	Remarks.
	No. of Wires.	Diam. of Wires.	No. of Wires.	Diam. of Wires.	No. of Wires	Diam. of Wires.	No. of Wires.	Diam. of Wires.				
Heavy Lifts, .	3	·078"	9	·084"	15	·084"	18	·101"	45	7	315	Six strands are of patent steel, and are wound round a central strand of soft steel.
Light Lifts, .	1	·060"	6	·060"	12	·060"	18	·060"	37	7	259	

The factors of safety allowed in the different parts are :—

Main framing, jib, tension rods, &c., ...	6
Wire ropes, ...	8
Centre holding down bolts, to allow for rusting, .	12

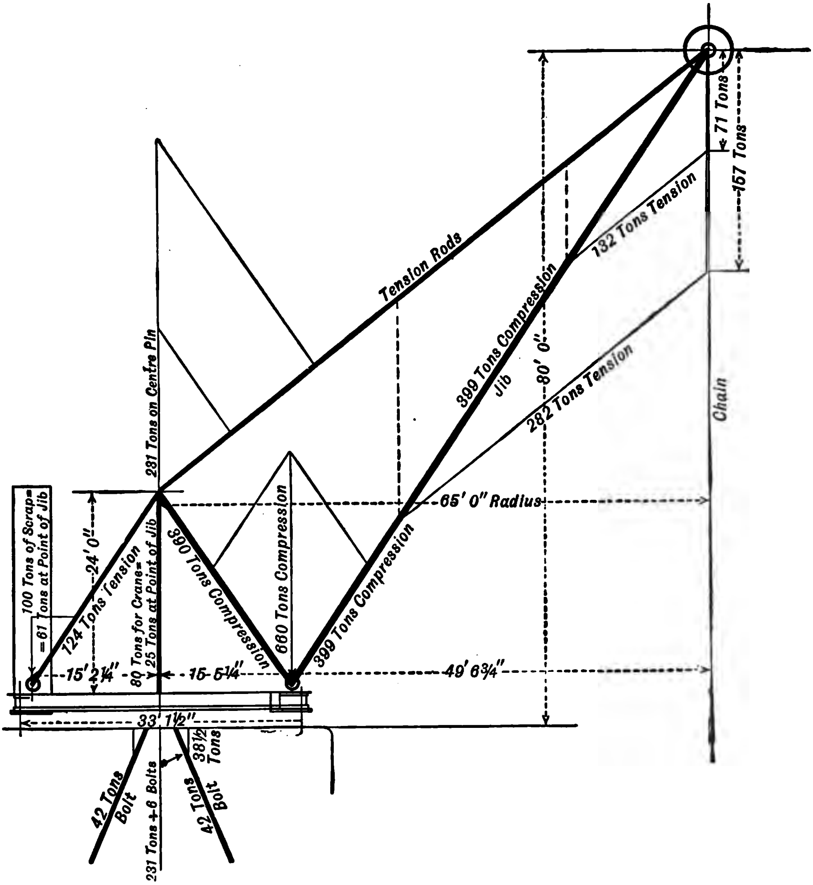


FIG. 14.—STRESS DIAGRAM FOR 130-TON CRANE. (TEST LOAD, 150 TONS.)

The accompanying figures show the Stress Diagram for the crane when loaded with 150 tons, or, including the weight of the gin block, 157 tons in all, and also for a gross load of 71 tons.

Fig. 14 is the Stress Diagram as worked out by the makers. Fig. 15*a* represents the Frame Diagram and Fig. 15*b* the Stress Diagram worked out for the Test Load as we have treated the previous cranes—viz., all in one figure.

First, find the values of the Forces D E and A B in Fig. 15*a* as already explained. Then begin the Stress Diagram with the force A B and follow with the forces B C, C D, D E, E F and F G. Now E L and G L fix the point L; B H and C H the point H; H K and L K the point K; K N and G N the point N; and N M and A M the point M.

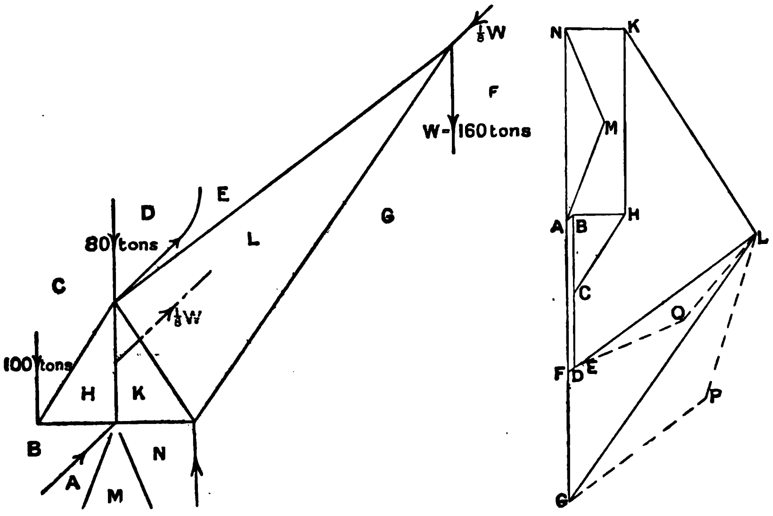


Fig. 15*a*.—Frame Diagram. Fig. 15*b*.—Stress Diagram.

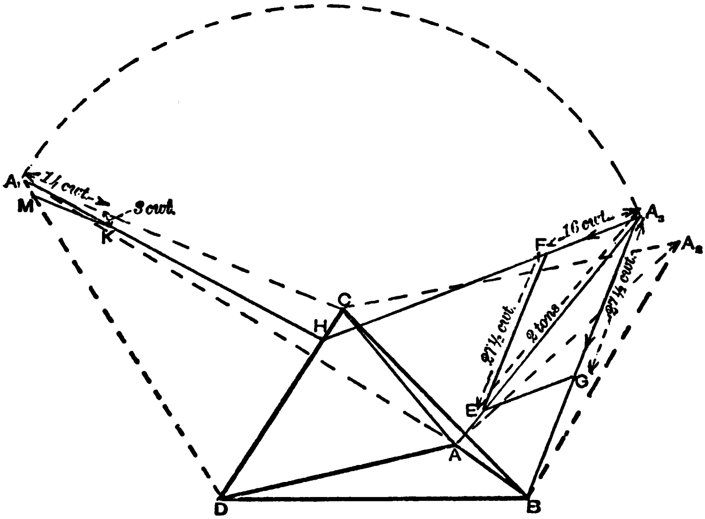
130-Ton Crane Worked out as in the Previous Examples.

The stress E L is borne by the two tie-rods and L O and E O represent the stresses in each tie-rod. The stress G L is divided over the two jibs, consequently L P and G P represent the stresses in each jib, as already explained.

The stress A M is divided over half of the holding down bolts and N M over the other half. The point about which the crane tends to topple over is the joint K L G N.

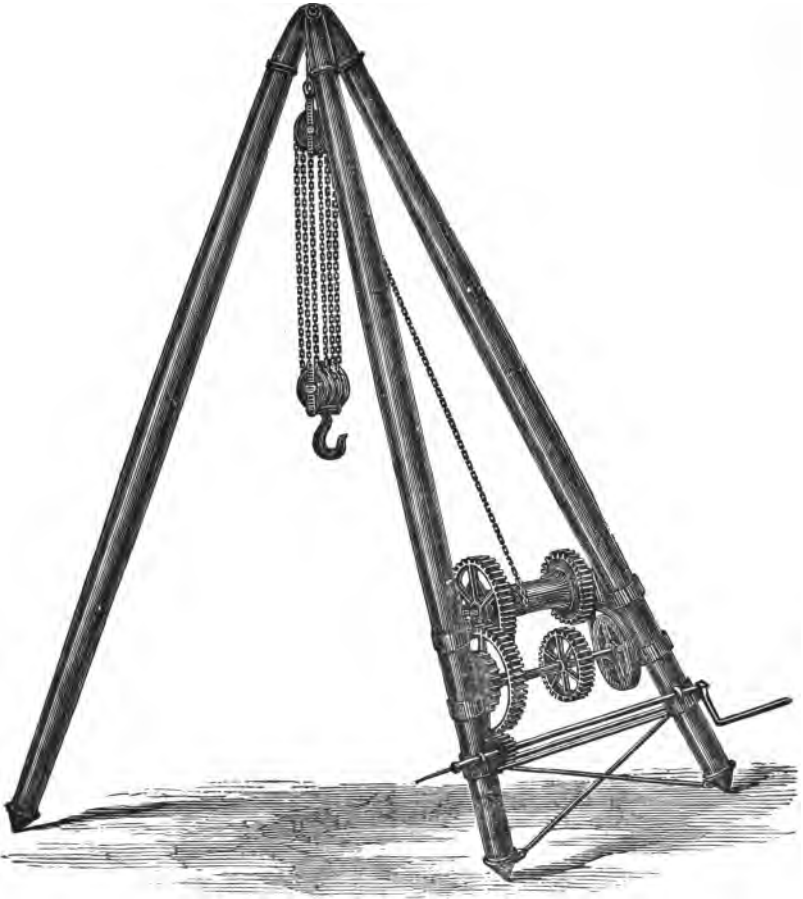
Example III.—A tripod whose vertex is A, and whose legs are A B, A C, A D, of lengths 8, 9, and 10 feet respectively, sustains a load of 2 tons. The ends B, C, D form a triangle, whose sides are B C = 7 feet, C D = 6 feet, B D = 8 feet, find by graphical construction the compressive stress in each leg. (S. and A. Exam., 1889.)

ANSWER.—Hitherto we have dealt only with forces in one plane, or symmetrical with respect to one plane. Thus, in the case of the sheer legs, in determining the compression on the front legs, we first found what would be the compression on an intermediate leg in the same plane as the two replaced, and equally inclined to both. This hypothetical leg would be in a vertical plane containing the back leg and the externally applied force, and evidently the stress in the back leg would not be affected by the substitution. In the present example we must find the corresponding intermediate leg, so that if the load be supported by it and the third leg, the stress in the latter will not be altered. It will be in a vertical plane containing the third leg, and will be represented by the line of intersection of this plane with that of the other two legs, hence the following solution:—Draw B C D the triangle formed by the feet of the tripod; to find the plan of the vertex, draw the triangle D C A_1, making C A_1 and D A_1 equal to C A and D A respectively. Similarly, draw the triangle A_2 C B; then A_1 A drawn perpendicular to C D, and A_2 A drawn perpendicular to C B, will give by their



STRESSES IN A TRIPOD.

intersection the plan of the vertex A, and we may now complete the plan. The vertical plane containing the leg A B will cut the line D C in a point H lying on the line B A produced;

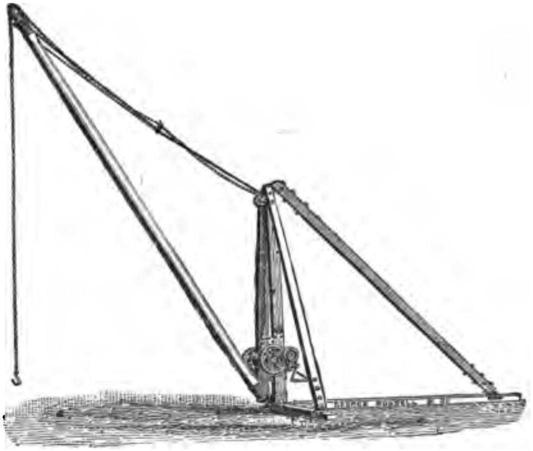


TRIPOD FOR LAYING PIPES.

then A_1 H will be the length of the required intermediate leg. Draw A A_3 at right angles to B A H, and make H A_3 = H A_1. Thus we get A_3 H, A_3 B, and the applied force A_3 E in one plane. The stress in the leg A B will be represented by $A_3 G_3$ and the resultant stress in the legs A C and A D—*i.e.*, the stres in the hypothetical leg A H by A F. Dividing this between the actual legs by the triangle of forces A_1 M K, where A_1 K = A F, we get the stress in A C represented by [illegible] K, and that in A D by A_1 M. The respective values are marked on the diagram.

As a practical example of the use of a tripod, we illustrate a form much used—in laying water and other mains—for lowering large pipes into position. Two of the legs are braced together, and carry a winch which may be used in conjunction with a block and tackle. The pipe is rolled on to wooden beams laid across the drain, and the tripod then placed in position over it. The pipe is slightly raised by means of the winch, and the wooden beams removed, when the pipe may be lowered with ease.

We also illustrate a simple hand crane as used by contractors for building, &c. It is of the kind discussed in connection with Fig. 7.



HAND DERRICK CRANE.

LECTURE XXVII.—QUESTIONS.

1. In a model of a crane the jib is 3½ feet long, the tie-rod is 3 feet long, and is fastened to a point 1 foot vertically above the lower end of the jib, What is the thrust on the jib when a weight of 20 lbs. is hung at the upper end of it? *Ans.* 70 lbs.

2. In a wharf crane the post, tie-rod, and jib measure 15, 20, and 30 feet respectively, what would be the nature and amount of the stresses in each of the three members when a load of 7 tons is suspended over the pulley at the jib head, (1) when the lifting chain passes from the pulley to the drum or barrel parallel with the jib, (2) when the drum is placed so that the chain passes from the jib head parallel with the tie-rod? (S. and A. Adv. Exam., 1890.)

3. In a hydraulic wharf crane the height of the post is 6 feet, the jib is 22 feet, and the tie-rod is 18 feet; find the horizontal thrust on the post when 5 tons are supported. In what way is the friction which opposes the *slewing* motion reduced to a minimum? *Ans.* 12·24 tons.

4. Find, either graphically or otherwise, the stresses on the jib and tie-bar respectively of a crane, whose jib measures 20 feet in length, when the tie-bar and post are 16 feet and 6 feet in length respectively, and a weight of 25 cwts. is suspended from the end of the jib. The line of direction of the chain after leaving the barrel or drum runs parallel to the tie-bar. Also calculate the pressure on the end of the handle of 16 inches radius when the weight is lifted, supposing the drum of the crane to be 15 inches in diameter, and the gearing to consist of a pinion of 12 teeth, gearing into a wheel with 72 teeth, while a second pinion of 18 teeth gears with a wheel of 56 teeth. (S. and A. Adv. Exam., 1893.)

5. A contractor's portable hand crane has a vertical post A B, to which the jib A C is inclined at 45°, and the tension rod B C makes with A B an angle A B C of 120°. The back stay from the head of the post B to the extremity D of the horizontal strut A D is inclined at an angle of 45° to A D. Find the weight of the counterbalance required at D to balance a load of 10 tons suspended from the end C of the jib. Determine also the nature and amount of the stress on the jib A C, and in the rods B C and B D? (The tension in the chain may be neglected.) (S. and A. Adv. Exam., 1896.)

6. A jib foundry crane consists of a vertical post A B, 16 feet long, fitted with pins working in sockets at both A and B. From the upper end A of the post extends a horizontal member A D, 28 feet in length, and from the foot B is a strut B C, which meets A D at a point 16 feet from A. A load of 20 tons being suspended from D, find the shearing stress on the pin at A, and the stress along the strut B C. (S. and A. Adv. Exam., 1892.)

7. A sheer-legs is formed of two sheer-poles B C, D C, each 25 feet in length, and secured to a base-plate in the ground at B and D. The wire guy or tension rope A C is attached to the ground at a point A, which is 60 feet distant from B D. The perpendicular from the top C of the poles meets the ground at a distance of 10 feet from the centre of the line B D, which is 15 feet long. Find by measurement or otherwise the tension in tons on the guy when a weight of 20 tons is suspended. (S. and A. Adv. Exam., 1888.) *Ans.* 11·3 tons.

LECTURE XXVIII.

CONTENTS.—Reactions on a Beam—First Method—Resultant of the Loads on a Beam—Reactions on a Beam—Second Method—Fink Truss—Trapezoidal Truss of Three Panels—Trapezoidal Truss of Five Panels—Example I.—Warren Girder—Linville or N Girder—Lattice Girder—Redundant Frame—Five Bay Lattice Girder—Lattice Girder loaded at Top Joints—Bending Moment—Definition—Shearing Force—Definition—Cantilever Uniformly Loaded—Examples II. and III.—Centre of Gravity of an Area—Moment of Inertia of an Area—Proof—Engine Mechanism—Questions.

Reactions on a Beam.—FIRST METHOD.—*First Case.*—In the Frame Diagram, Fig. 1*a*, we have a beam with five concentrated loads upon it. The line of action and point of application of the left-hand reaction, and the point of application of the right-hand reaction are known.

We commence Fig. 1*b* by drawing the line of loads, B C, C D, D E, E F and F G. Then, any point O is taken and joined with all the points in the line of loads as illustrated in the figure. This point O is called a Pole and Fig. 1*b* a Polar Diagram.

Since the point of application of the right-hand reaction (Fig. 1*a*) is the only one of its elements which is known, we must begin at this point 1 when drawing the Funicular Polygon 1 2 3 4 5. In the Polar Diagram, Fig. 1*b*, the line O F comes between the loads E F and F G. Then, from the point 1, in the Frame Diagram, draw the line 1—2, parallel to O F. The line 1—2 begins in the line of action of the load F G and ends in the line of action of the load E F—viz., the two loads between which the line O F lies in the Polar Diagram. Between the lines of action of the loads E F and D E, draw the line 2—3 parallel to the line O E in the Polar Diagram, which lies between these same two loads. Similarly draw 3—4 parallel to O D and 4—5 parallel to O C. On joining the point 5 with the point 1 we close the Funicular Polygon. This closing line 5—1 has one end 5, in the line of action of the reaction A B, and the other end 1, in the line of action of the reaction A G. Therefore, if a line O A be drawn parallel to this closing line from O, it must lie between the reactions in the line of loads, just as the other lines radiating from O have done.

If we draw B A from B in the line of loads parallel to the line of action of the reaction A B, so as to meet the line O A in the point A, then A joined with G completes the external force polygon.

Second Case.—If the two ends of the beam had been anchored, then the lines of action of the reactions G A and A B would have been parallel to each other and to the line joining B with G in the Load Diagram. In this case the lines of action of the reactions would be drawn on Fig. 1*a*, and the point 1 taken anywhere in the line of the reaction G A; or we could begin with the point 5 anywhere in the line of the reaction A B, and draw the Funicular Polygon as stated above.

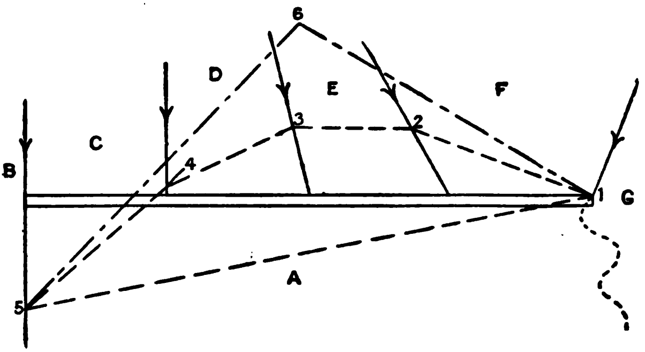


FIG. 1*a*.—FRAME DIAGRAM AND FUNICULAR POLYGON.

If the load F G had not existed, we would still have begun at the point 1 for the first case. The line O F would then be between the load E F and the reaction F A. In the second case mentioned above, the solution would still be the same as before.

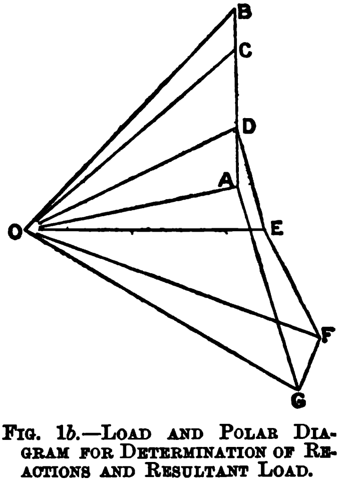


FIG. 1*b*.—LOAD AND POLAR DIAGRAM FOR DETERMINATION OF REACTIONS AND RESULTANT LOAD.

Resultant of the Loads on a Beam.—From the point 1 in the Frame Diagram, Fig. 1*a*, draw the line 1—6 parallel to the line O G in the Polar Diagram. This line O G lies between the load F G and the resultant G B. Now draw the line 5—6 parallel to the line O B, so as to intersect the line 1—6 in the point 6. Then the point 6 is a point in the line of action of the resultant of the loads. If through the point 6 we draw a line parallel to B G, then that line will represent the line of action of the resultant load. This line cuts the beam at the point round which the beam would balance. The magnitude of the resultant is represented by the length of the line B G.

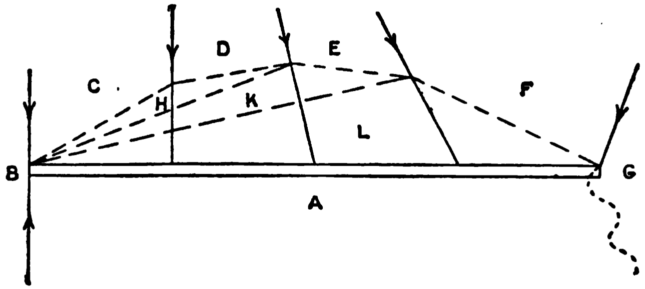


FIG. 2*a*.—FRAME DIAGRAM.

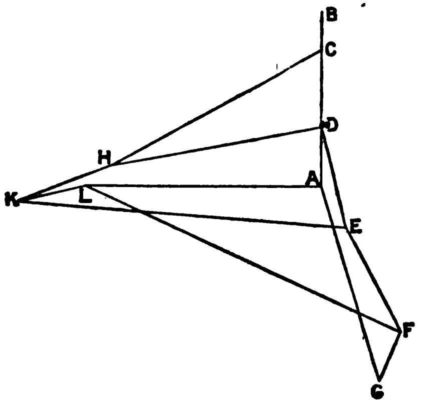


FIG. 2*b*.—LOAD AND REACTION DIAGRAM FOR DETERMINATION OF REACTIONS.

If we wished to find a point in the line of action of the resultant of the loads C D, D E, and E F, then from the point 2, draw a line parallel to the line O F, and from the point 4 a line parallel to the line O C, so as to intersect the first line from the point 2. This point of intersection is a point in the line of action of the resultant of C D, D E, and E F. Its line of action passes through this point and is parallel to the line joining C with F, while its magnitude is represented by the line C F.

Reactions on a Beam.—SECOND METHOD.—On the top of the beam draw a Firm Frame of any shape, having its joints in the lines of action of the loads as indicated in the Frame Diagram, Fig. 2*a*. Letter the Frame Diagram according to

Bow's method, and draw the Load and Stress Diagram as already explained. As a graphic solution the second method is the more accurate of the two, and may be applied with greater ease in the case of lattice girders, for we have only to join the points of the girder with one end, or some points with one end, and the remainder with the other end.

Fink Truss.—This truss is largely used in America for wooden bridges and is represented in the Frame Diagram, Fig. 3*a*. It consists in this case, of a primary truss B A G, and two secondary trusses. The divisions of the beam are called panels. The truss in Fig. 3*a* is therefore a Fink Truss of four panels.

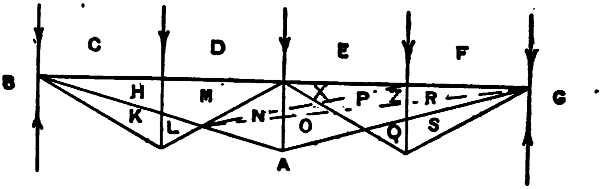


FIG. 3*a*.—FRAME DIAGRAM, FOR A FINK TRUSS.

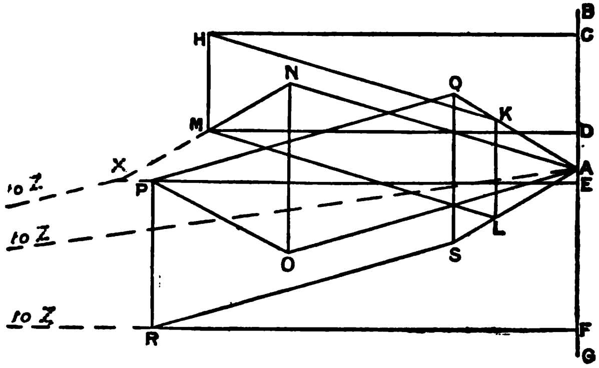


FIG. 3*b*.—STRESS DIAGRAM.

If the foot of every upright had been joined with each end of the beam instead of as shown, the truss would be called the **Bollman Truss**, of which the solution is similar to the following explanation for the Fink Truss. Draw the external force polygon as explained for Figs. 1*a* and 2*a*.

FIRST METHOD FOR STRESS DIAGRAM.—Join the joint M N A L with the joints E F R P and F G A S R, as shown by the dotted lines in the Frame Diagram, Fig. 3*a*. Call the spaces thus formed

X and Z as shown in the Frame Diagram. This will enable us to determine the point M in the Stress Diagram, Fig. 3*b*. Draw F Z and A Z, parallel to the bars F Z and A Z. This gives the point Z. Then Z X and E X fix the point X; and X M and D M fix the point M. The Stress Diagram may now be completed in the usual way, and the closing line will form a check line.

SECOND METHOD FOR STRESS DIAGRAM.—In Fig. 4, we have four forces in equilibrium acting at a point and their lines of action lie along two straight lines. The condition for equilibrium is, that H M is equal and opposite to K L and M L is equal and opposite to K H. The Force Polygon will therefore be a parallelogram and may be represented by H M L K in the Stress Diagram, Fig. 3*b*.

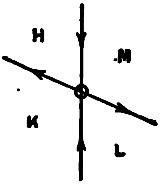


FIG. 4.—FORCES ACTING IN STRAIGHT LINES AT A POINT.

From the above, it is evident that the stress in H M must be equal to the load C D and the stress in K L equal to the stress in H M. Therefore, draw A K and A L parallel to the bars A K and A L respectively and draw K L between them parallel and equal in length to C D. This will determine the points K and L in the Stress Diagram, Fig. 3*b*, which may now be completed. The closing line will form a check as in the first method.

Trapezoidal Truss of Three Panels.—In this Truss, Fig. 5*a*, the space F will always remain a parallelogram, as was the case in the Queen Post Frame which has been already discussed. In fact this is a similar case to the roof considered with a continuous tie-beam.

If the beam is uniformly rigid, then whatever distance the joint C D G F is deflected below the horizontal line, the joint B C F E rises an equal distance above the horizontal line. Therefore, the difference between the vertical component of the load C D and the stress action in G F, must be equal and opposite to the difference between the vertical component of the load B C and the stress action in E F.

In order to obtain the Stress Diagram, Fig. 5*b*, we first of all draw the external force polygon B C D A, and determine the point A, as explained for Figs. 1*b* or 2*b*. Then draw A F parallel to the bar A F and B_1 C_1 D_1 perpendicular to A F. B B_1, C C_1 and D D_1 are parallel to A F. B_1 C_1 and C_1 D_1 will be the vertical components of B C and C D respectively.

Since C_1 D_1 is greater than B_1 C_1, the joint C D G F will be deflected downwards. Therefore, F G will be less than C_1 D_1

and is as much less than $B_1 C_1$ as $C_1 D_1$ is less than E F. This is the same thing as saying that G F and F E are together equal to $B_1 D_1$.

The point E lies on the line A E drawn parallel to the bar A E and similarly for the point G. Place the line E G between the lines A E and A G, parallel to the bars G F and F E and equal in length to $B_1 D_1$. This determines the points E and G. Where E G cuts the line A F fixes the point F. Then by joining F C, E B and G D we complete the Stress Diagram.

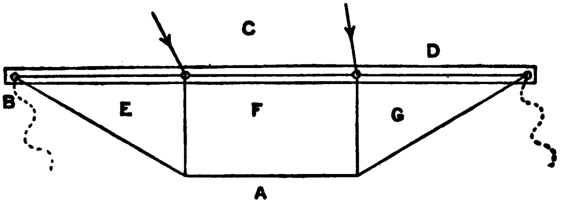


Fig. 5a.—Frame Diagram for a Trapezoidal Truss.

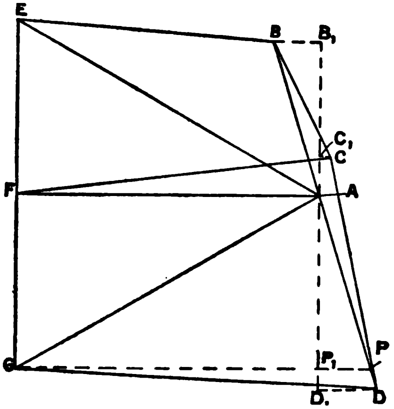


Fig. 5b.—Stress Diagram.

Another method of determining the point G is to make A P_1 equal to one-half of $B_1 D_1$, or A P equal to one-half of B D, and draw P_1 G or P G parallel to A F, so as to cut the line A G. This determines the point G.

Trapezoidal Truss of Five Panels.—In this truss, Fig. 6*a*, the space O always remains a parallelogram, but the other spaces cannot change. Similar reasoning to what we used in the previous figure will show that the loads D E and E F are together equal and opposite to the sum of the stress actions P O and O N. The load C D is equal to the stress in L N, and also equal to the stress in K M. Similarly the load F G is equal to the stresses in the bars P R and Q S. It follows from this, that the stresses R P, P O, O N, and N L are together equal and opposite to the loads C D, D E, E F, and F G. The Stress

Diagram, Fig. 6*b*, may be determined by drawing A K parallel to the bar A K, and making K M equal to O D. Then through M draw M N parallel to the bar M N. The point N lies on this line. Draw A S parallel to the bar A S, and make S Q equal in length to F G. Through Q draw Q P parallel to the bar Q P. The point P lies on this line. Now O P and O N are together equal to F E and E D. Consequently, if we make P N equal to F D and parallel to the bars O P and O N, this will determine the points P and N. Draw A O parallel to the bar A O. This fixes the point O. The points L and R may now be determined, and the finishing lines of the diagram inserted.

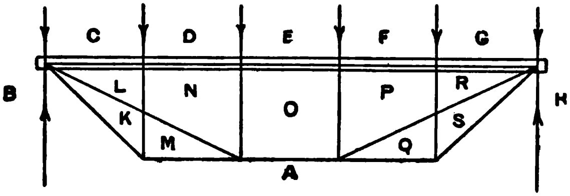


FIG. 6*a*.—FRAME DIAGRAM FOR FIVE PANEL TRAPEZOIDAL TRUSS.

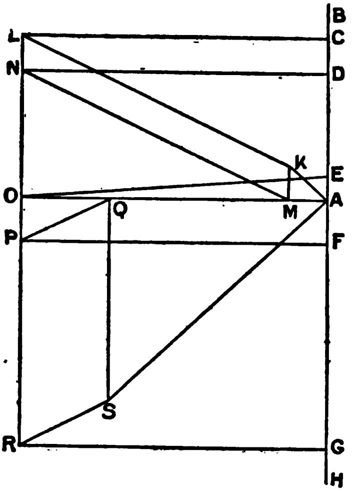


FIG. 6*b*.—STRESS DIAGRAM.

In Fig. 7 the forces acting on the left-hand portion of the beam are shown as taken from the Stress Diagram, Fig. 6*b*. They are the actions of the pins on the beam, and of the right-hand half on that end.

In Fig. 8 is shown the form which the left-hand end must take from the previous assumptions. The three points shown in a straight line must always remain in a straight line. There will be three points of contrary flexure in the length of the beam.

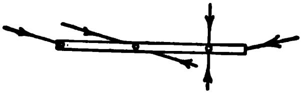


Fig. 7.—Forces acting on the Left-Hand Half of the Beam.

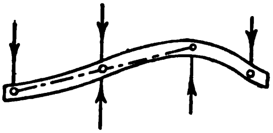
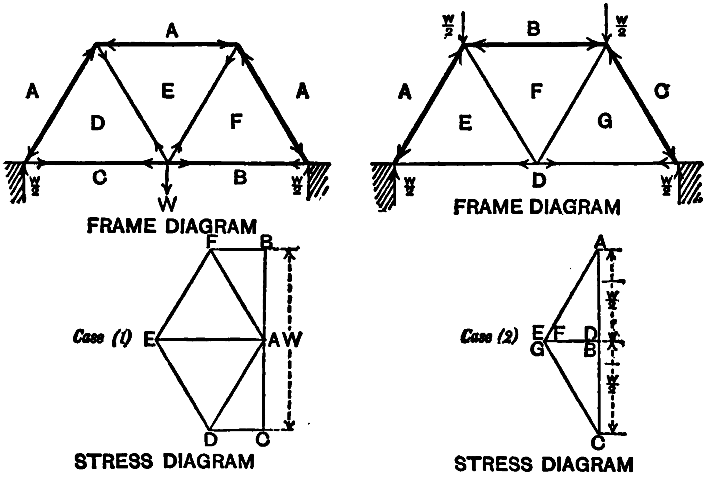


Fig. 8.—Form into which the Forces Bend the Left-Hand Half of the Beam.

These Trapezoidal Trusses are deficient frames made redundant by stiff joints.

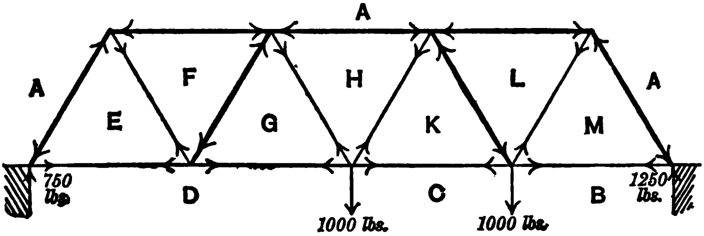


Triangular Frame.

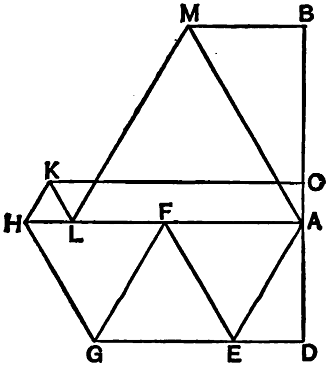
Example I.—A triangular frame, consisting of three equilateral triangles, is loaded with a weight W. Find the stresses on the several members of the frame (1) when W is hung at the lower apex of the central triangle, (2) when each of the triangles is loaded at the upper apex with $\frac{W}{2}$.

Answer.—Since the loading is symmetrical in both cases, the reactions are each equal to one-half of the total load—that is, to $\frac{1}{2}$ W.

Case (1).—Draw B C vertical and equal to W units. Bisect it at A, so that C A and A B are the reactions. Make C D parallel to the bar C D, and A D to A D. This gives us D. Then draw D E and A E respectively parallel to the bars D E and A E, so as to obtain E. A F and E F fix F, and B F completes the Stress Diagram. B F should be parallel to the bar B F, and this gives us a check on our work.



FRAME DIAGRAM



STRESS DIAGRAM

Fig. 9.—Warren Girder.

Case (2).—Here we must make A B and B C each equal to $\frac{1}{4}$ W. D A and C D will represent the reactions, D being the middle point of B C, and therefore coincident with B. A E and D E determine E, and C G and D G give G. E and G will coincide since B and D do, and everything is symmetrical. The point F also coincides with E and G, so that there is no stress in the bars E F and F G.

Warren Girder.—Fig. 9 illustrates a Warren Girder of four bays or panels, which is simply an extension of the above triangular frame. The upper horizontal member is called the **Upper Boom** or **Flange**, and the lower horizontal member is called the **Lower Boom** or **Flange**. The inclined members are called **Lattice Bars** or **Braces**. The joint A F E or E F G D is called an **Apex**. The angle of triangulation is usually 60°, but sometimes the triangles are right-angled isosceles. When loads are applied at the centres of the lower members, tie-rods are put from the centres of the lower members to the opposite apices. These tie-rods transmit the central loads to the upper apices. The Frame Diagram would then show a system of loads at the lower and upper apices. From what has already been said the student should find no difficulty in drawing the Stress Diagram, which is obtained in exactly the same way as those in Example I.

Linville or N Girder.—This girder differs from the Warren Girder, in that the bars connecting the two horizontal booms are placed alternately vertical and oblique, forming a series of right-angled triangles instead of the corresponding equilateral ones in the Warren Girder. It is so arranged that the shorter vertical bars shall be in compression and the oblique ones in tension. This clearly tends to a saving of material and a diminution of weight, since compression members, unless very short, must, other things being equal, be much heavier than tension members. Also, compression members are much strengthened by shortening, while a tension member is not weakened by lengthening.

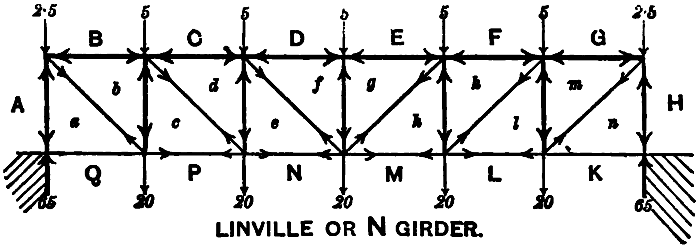
We may determine the reactions by a "substituted frame" by the Funicular Polygon, or, in this case, by calculation.

In drawing the Stress Diagram we observe that A *a* must be equal to Q A, and *a* Q is zero, since the reaction is vertical. The members *a* Q and *n* K are necessary to give the required stability. These, together with the end vertical members, might be dispensed with by carrying the supports up to the upper boom.

Draw the lines representing the loads and reactions, viz.:— K L, L M, M N, N P, P Q, Q A, A B, B C, C D, D E, E F, F G, G H, and H K. Since A *a* = A Q, and H *n* = H K, so that *a* coincides with Q, and *n* with K, we may draw the Stress Diagram as follows :—

B *b* and *a b* fix the point *b*, and G *m* and *n m* the point *m*.
P *c* „ *b c* „ *c*, „ L *l* „ *m l* „ *l*.
C *d* „ *c d* „ *d*, „ F *k* „ *l k* „ *k*.
N *e* „ *d e* „ *e*, „ M *h* „ *k h* „ *h*.
D *f* „ *e f* „ *f*, „ E *g* „ *h g* „ *g*.

By then joining fg we complete the Stress Diagram, and this line should be parallel to the central strut.



LINVILLE OR N GIRDER.

FIG. 10*a*.

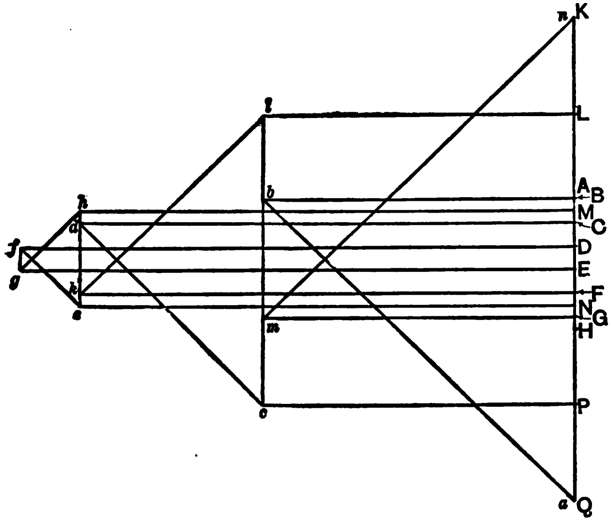


FIG. 10*b*.—STRESS DIAGRAM.

Lattice Girder.—Although in Fig. 12, we have drawn the external force polygon for Fig. 11, we can go no further with the Stress Diagram for frames of this kind until we have determined in some way the stress in one member, because these are redundant frames.*

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* See the beginning of Lecture XXIV. for the properties of redundant frames.

ASSUMPTION.—**We assume, first of all, that all the members have been accurately fitted; that is, the frame is not initially stressed.**

One method of solution which has often been suggested is, to assume that the shear over any section is taken equally by the braces in that section. We shall however prove that this latter assumption is not consistent with the actual conditions. The shear on the right-hand bay of Fig. 11 is evidently one-half of B C, and according to the above method, the vertical component of the stress in the member G H would be equal to one half of the shear, that is one quarter of B C, whilst the vertical component of the stress in the member G K would be equal to the other half of the shear, that is also one quarter of B C. Similarly, for the left-hand bay, according to the above assumption, the vertical components of the stresses in E G and F G would each be one quarter of B C.

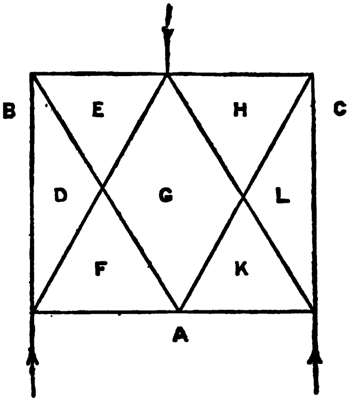

Fig. 11.—Frame Diagram.

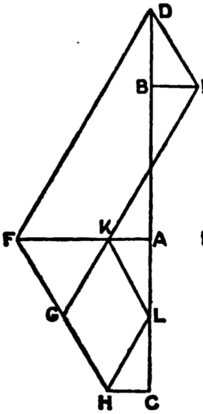

Fig. 12.—Stress Diagram with Unequal Stesses in Braces.

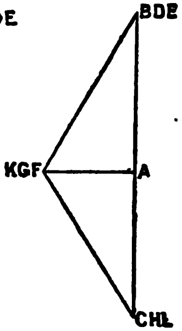

Fig. 13.—Stress Diagram with Equal Stresses in Braces.

Two Bay Lattice Girder.

Let us suppose the vertical component of the stress in G H and K L, to be one-quarter of B C, and then draw the Stress Diagram, Fig. 12. There A L is equal in length to the vertical component of K L and also equal to one-quarter of B C. The Stress Diagram may now be completed in the usual way. It shows that if the stress in G H has a vertical component of one-quarter of B C, then the vertical component of the stress in E G is three-quarters of B C. This is a consistent result, because three-quarters and one-quarter of B C acting vertically together at the joint E B C H G will balance B C. But, if the members have been properly put together, there is no reason at all why the stress in E G should be greater than the stress in G H. The most sensible assumption to make is, that the members E G and G H will each have a stress of which the vertical component is one-half of B C. The Stress Diagram for this assumption is worked out in Fig. 13.

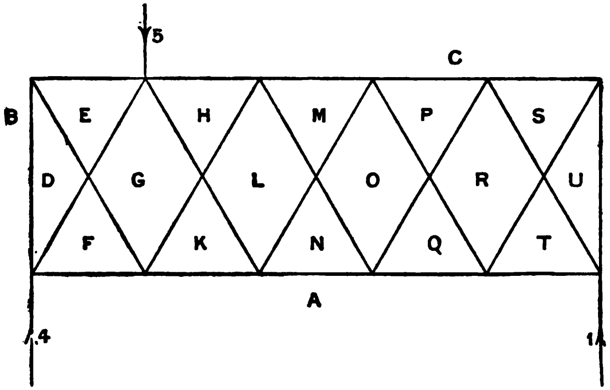

Fig. 14a.—Frame Diagram for Five-Bay Lattice Girder.

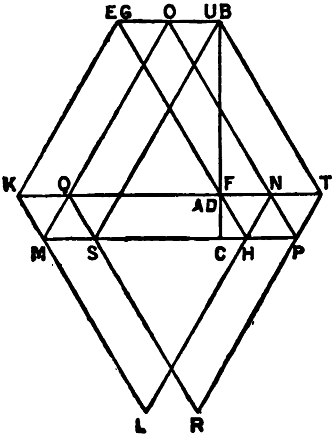

Fig. 14b.—Stress Diagram. When Braces E G and D F are Removed.

Lattice Girder of Five Bays.—To Determine what Proportion of a Single Load is Transmitted along each Brace.—In the Frame Diagram, Fig. 14*a*, we have assumed a load of 5 units at the joint B C H G E. From inspection, it will be evident that the reaction A B will be 4 units and A C will be 1 unit. The reactions may be determined by a substituted triangular frame with its vertex at the loaded joint and rafter ends at the abutments. The Stress Diagram, Fig. 14*b*, has been drawn on the assumption that the braces E G and D F have been withdrawn. If the bar D F is removed, then the spaces D and F will have only one letter, let this be called D. Then B D and A D fix the point D; D F and A F the point F; D E and B E the point E; E G and F G the point G and so on, until the Stress Diagram is completed.

From the Stress Diagram, Fig. 14*b*, we see that the vertical components of the stresses in the braces* G H, K L, L N, M O, O P, Q R, R T, S U and U C are each equal to the load B C, and that the vertical components of the stresses in the remaining braces T U, R S, on to B D are each equal to four units; also that the kind of stress alternates between a push and a pull throughout the braces. The vertical component of the push in the lattice bar G H, is balanced by the vertical component of the pull in L N. Also, the vertical component of the pull in M O, is balanced by the vertical component of the push in O P, and similarly for Q R and R T, S U and U C. Now, the push in U C is balanced by the reaction C A, combined with the vertical component of the pull in T U, and since the reaction C A is one unit the vertical component of the pull in T U must be four units. This four units of vertical component of the pull in T U, is transmitted through the remaining lattice bars as a push and a pull alternately, until it reaches B D as a push, and is there balanced by the reaction A B.

The stress in each lattice bar produces a strain that will cause the load to dip. Now, since the stresses are severe and every lattice bar is stressed, the removal of the brace E G produces a very yielding frame.

In Fig. 15 we have the Stress Diagram which is obtained from the Frame Diagram, Fig. 14*a*, by removing the brace G H. The removal of this brace means, that there will be no stress in the bars L N, M O, O P, Q R, R T and S U. Since there is no stress in S U there can be none in C S or C U. Therefore, S and U will coincide with C and the Stress Diagram can now be completed.

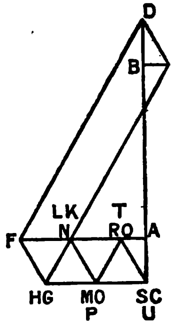

Fig. 15. — STRESS DIAGRAM WITH BRACE G H REMOVED.

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* Since the two members G H and K L, Fig. 14*a*, are parts of the same lattice bar or brace and the stresses in them are the same in every element, we may refer to them as the lattice bar or brace G H or K L and similarly for the others.

Fig. 15 shows that the lattice bar E G is the only one having the vertical component of its stress equal to B C. Also that each of the bars B D, D E, G K, L M, O Q and R S has the vertical component of its stress equal to one unit of the load. Further, that the stresses are alternately push and pull and that the remaining lattice bars are not stressed.

The vertical component of the push in the lattice bar E G is balanced by the reaction A B, in combination with the stress in B D. Since the vertical component of the push in the brace E G must balance the load, its value is 5 units and the reaction A B has a value of 4 units; therefore, the stress in the bar B D must be a pull of 1 unit. This vertical component of the pull in B D is balanced by the vertical component of the push stress in D E and so on, until the vertical component of the push in T U is balanced by the reaction A C of one unit.

Since the stress is severe in only one lattice bar and a number of the bars are unstressed, the removal of the brace G H produces a very much less yielding frame than the removal of the brace E G.

When the frame is complete as shown in the Frame Diagram, Fig. 14*a*, the vertical components of the pushes in the lattice bars E G and G H must together carry the load. The amount which each carries will be inversely proportional to the yieldingness of the system of bars to which each is connected. The lattice bar E G will therefore carry more than the lattice bar G H.

In this case, we have made the following assumptions:—

(1) That the Frame is not initially stressed.

(2) That the vertical component of the push stress in the left-hand lattice bar meeting in a joint at which a load is applied is equal to that portion of the left-hand reaction which that load produces. Also, that a similar relation exists between the vertical component of the push stress in the right-hand lattice bar and the right-hand reaction.

Lattice Girder Loaded at Top Joints.—For the lattice girder, Fig. 16*a*, we must first determine the reactions. This may be done by one of the methods for Figs. 1 and 2, of which the latter one, is the better and simpler. By joining the lower left-hand corner with each of the upper joints we obtain a Simple Triangular Frame, and by drawing the Stress Diagram for it, we can determine the reactions. Since the frame is redundant we must first calculate the stress in one member before we can begin the Stress Diagram. The bar B K is the most suitable one.

From the second assumption we observe that:—

(1) The load B C, produces no stress in K L and a push stress of 3 units in B K.

(2) The load C D, produces a push stress having a vertical component equal to 4 units ($\frac{4}{5}$ C D) in K M. This component is entirely balanced by the reaction which the load C D produces in A B—viz., 4 units. This load produces therefore no stress in B K.

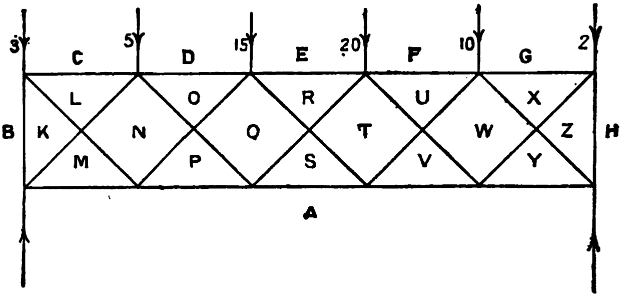


Fig. 16*a*.—Frame Diagram for a Loaded Lattice Girder.

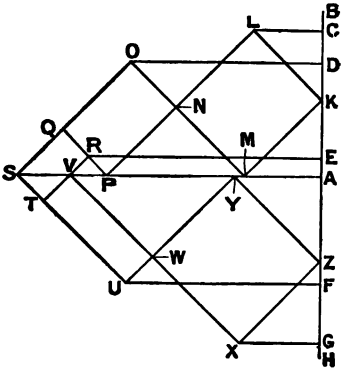


Fig. 16*b*.—Stress Diagram.

(3) The load D E, produces a push stress having a vertical component equal to 9 units ($\frac{3}{5}$ D E) in O Q. This stress induces a pull stress in M N having a vertical component equal to 9 units and this pull stress induces in B K a push stress of 9 units.

(4) The load E F, produces a push stress having a vertical component equal to 8 units ($\frac{2}{5}$ E F) in R T. This stress induces a pull in P Q, and a push in N L, the vertical component of which is balanced by the reaction produced in A B. This load produces therefore no stress in B K.

(5) The load F G, produces a push stress having a vertical component equal to 2 units ($\frac{1}{5}$ F G) in U W. This stress is transmitted as push and pull until it reaches B K as push and is balanced by the reaction A B.

(6) The load G H, produces no stress in B K.

Summing up the push stresses in B K we have :—

3 units due to B C,
0 units due to C D,
9 units due to D E,
0 units due to E F,
2 units due to F G,
0 units due to G H.

This gives a total push stress of 14 units in the bar B K.

From the point B, in the line of loads, Fig. 16*b*, draw a line B K parallel to the bar B K, and make the length of B K equal to 14 units to the same scale as the line of loads. This determines the point K in the Stress Diagram. Then K L and C L fix the point L and so on point by point until the Stress Diagram is finished. The finishing line forms a check line.

In calculating the stress in the lattice bars, we have sometimes a push stress and sometimes a pull stress. We must therefore pay due regard to the sign of the stress when adding up the various stresses. The force C D produces a push stress in the bar N O, having a vertical component of 1 unit, while E F produces a push stress in R T having a vertical component of 8 units. This push induces a pull in N O also having a vertical component of 8 units. Therefore, the resultant stress in N O is a pull having a vertical component of 7 units.

Referring to Fig. 16*a*, we may observe that sometimes, vertical ties are put in the spaces N, Q, T and W. The action of these ties is to prevent distortion of the rectangles in which they lie. Therefore unless these rectangles have become distorted, they will not be stressed. This distortion will depend upon the relative yieldingness of the two systems of bars forming the rectangle. In this example suppose a tie in the space N and a load applied at the lower joint. Then, before this load can stress the tie, the lower joint must come down more than the upper joint. The safest assumption to make is, that the ties carry no portion of the loads. Some lattice bridges have struts in the spaces N, Q, T and W of Fig. 16*a* and the lattice bars are all ties—that is, they are only able to stand a pull stress. On drawing the Stress Diagram for such a case, we must omit a bar if a push stress comes in it, and use the other tie.

One tie in each bay must carry the shear in that bay. This will enable us to calculate the pull in that tie. This is the best method of drawing the Stress Diagram for such cases.

Bending Moment.—DEFINITION.—**The Bending Moment at any point in a beam, is the algebraic sum of the moments with respect to that point, of all the external forces acting on the portion of the beam on either side of that point.**

In order to draw the Bending Moment Diagram of Fig. 17, we must proceed as if we were going to find the reactions, by means of the Funicular Polygon and Polar Diagram as explained for Fig. 1*a*.

The Funicular Polygon drawn in this way is a Bending Moment Diagram.—That is, if a vertical line be drawn from a point in the beam, to cut the bounding lines of the Funicular Polygon, the intercept on this line which lies between those bounding lines, represents to a certain scale the bending moment on the beam at that point.

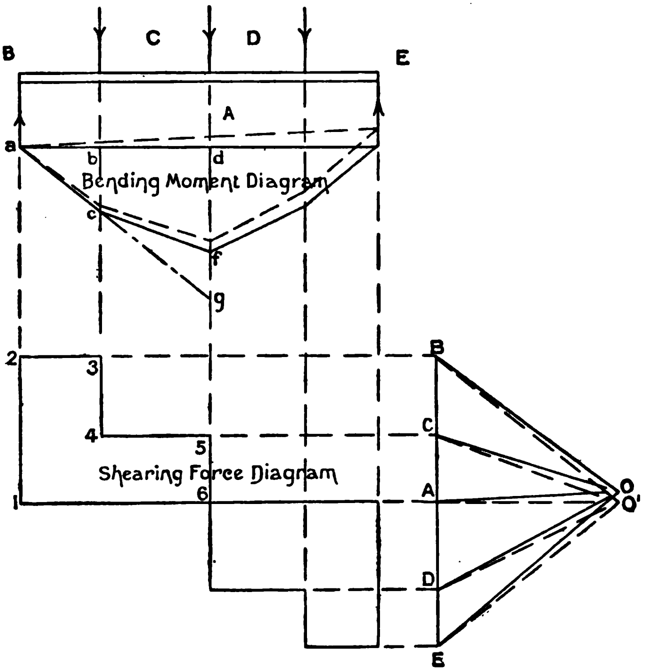


Fig. 17.—Bending Moment and Shearing Force on a Beam.

Having found the point A in the line of loads, by drawing A O′ horizontal and of any suitable length, we draw a Polar Diagram with this point O′ as the pole and the corresponding Funicular Polygon, when we obtain a Bending Moment Diagram on a horizontal base.

Proof.—The Bending Moment at the point where the load

B C acts, is equal to A B × *a b* Units of Moment. The Triangles *a b c* and O′ A B are similar, having the sides *a b*, *b c* and *c a* of the one respectively parallel to the sides O′ A, A B and B O′ of the other.

Therefore, $bc : ab :: BA : AO'$

Hence, $$bc = \frac{BA \times ab}{AO'}$$

Similarly, $$df = \frac{BA \times ad - BC \times bd}{AO'}.$$

That is, the number of units of length in *b c*, when measured with the scale for the load line, would give the Bending Moment, if O′ A measured 1 unit on the scale of length for the Beam.

SCALE FOR BENDING MOMENT DIAGRAM.—**Subdivide the unit of the scale used for the line of loads, into as many parts as the line O′ A contains the unit of the scale used for the length of the beam. Then, one of these subdivisions will be the unit for the Bending Moment scale.** It is found convenient to make O′ A ten units of the length scale.

Shearing Force.—DEFINITION.—**The Shearing Force on any transverse section of a beam is equal to the algebraic sum of all the external forces acting on the portion of the beam on either side of that section.**

In order to draw the **Shearing Force Diagram** of Fig. 17, no explanation is necessary, beyond following out the lines of the figure. The Shearing Force on any transverse section of the beam lying between the loads B C and C D, is, from the definition, equal to the force A B minus the force B C. Therefore, the length between the line 4—5 and the line 1—6 will measure the Shearing Force to the scale of the line of loads.

Cantilever Uniformly Loaded.—The cantilever shown in Fig. 18 may be considered as 12 feet long. The loads indicated are therefore equivalent to a uniform load per foot run. They act at the centre of each of the portions. By drawing from A, B, C, &c., on the Load Line, horizontal lines in the spaces A, B, C, &c., as shown, we determine the Shearing Force Diagram. If we divide the beam into smaller divisions and draw the Shearing Force Diagram, the stepped line will become more nearly a straight line. Consequently, when divided into infinitely small parts, the Shearing Force Diagram becomes the Triangle R P Q. The length of the line Q R is the total load on the Beam.

The Bending Moment Diagram of Fig. 18 is determined by drawing in the spaces A, B, C, &c., lines parallel to the lines O A, O B, O C, &c. The limiting form of the curve Q S will be a parabola with its vertex at S, and the value of the length Q T will be the Maximum Bending Moment.

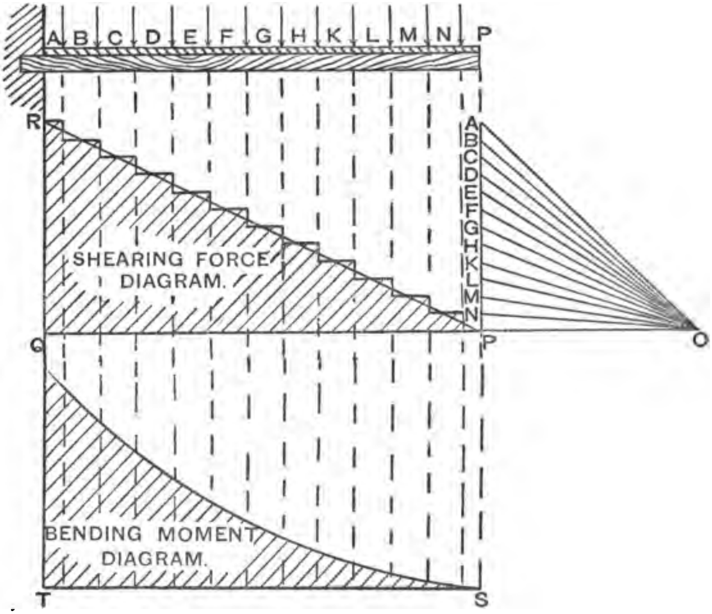


FIG. 18.—UNIFORMLY LOADED CANTILEVER.

Beam Uniformly Loaded and with Concentrated Loads.—Draw, as already explained, the Shearing Force Diagram for the concentrated loads. This is H A *1 2 3*, &c., on Fig. 19. Set off H P and K Q, each equal to half the total uniform load on the beam, and join P with Q. Then H P Q K is the Shearing Force Diagram for the Uniform Load. Adding the ordinates of the two diagrams together we derive the Combined Shearing Force Diagram H *a b c d e 5 6 f*, &c., of Fig. 19.

Draw the Bending Moment Diagram (L *n* M, Fig. 19) for the concentrated loads as described for Fig. 17. Then draw on the opposite side of L M a parabola, having its axis bisecting L M at right angles, and the ordinate at the centre of L M equal to the Maximum Bending Moment due to the uniform load. This ordinate must be measured to the same scale as that of the ordinates of the concentrated Bending Moment Curve. The combined ordinate measures the Combined Bending Moment.

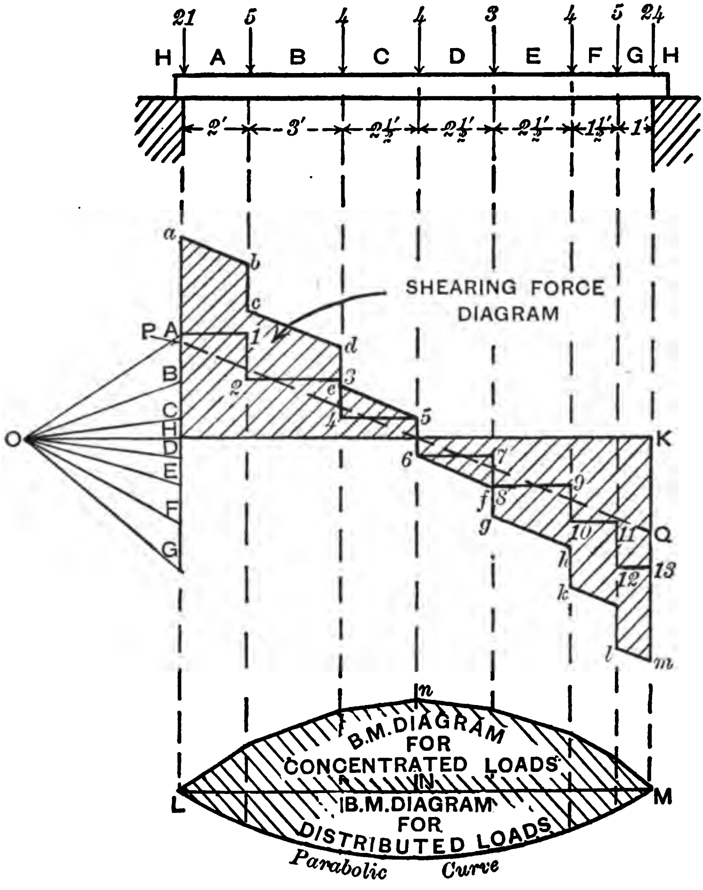
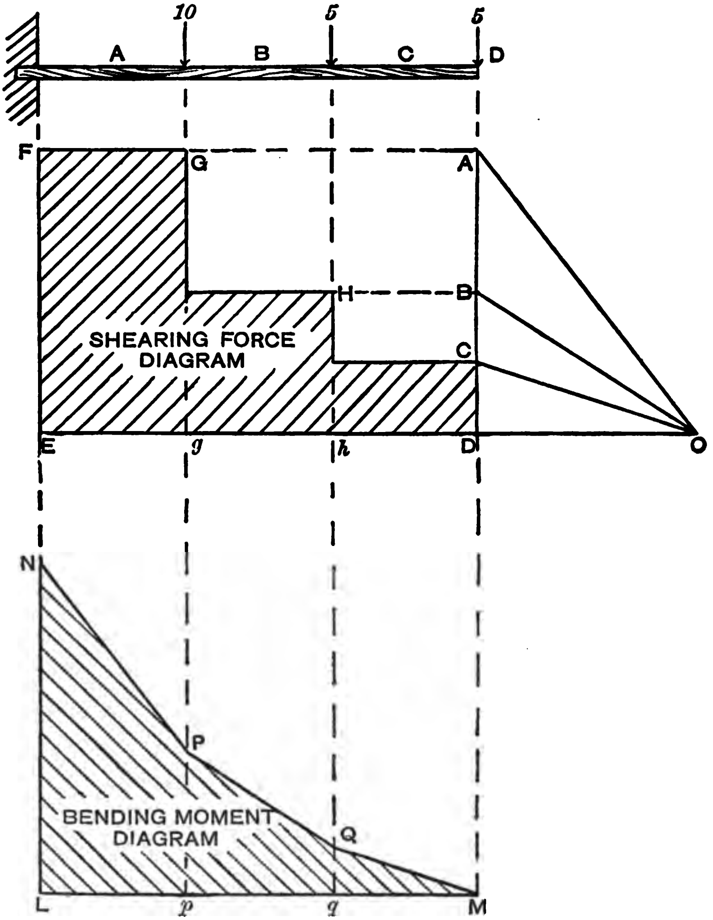


FIG. 19.—BEAM WITH UNIFORM AND CONCENTRATED LOADS.

EXAMPLE II.—A cantilever 15 feet long has a load of 5 tons at its outer end, 5 tons at 5 feet from it, and 10 tons at a point 10 feet from the end. Find graphically the diagrams of shearing force and bending moment.

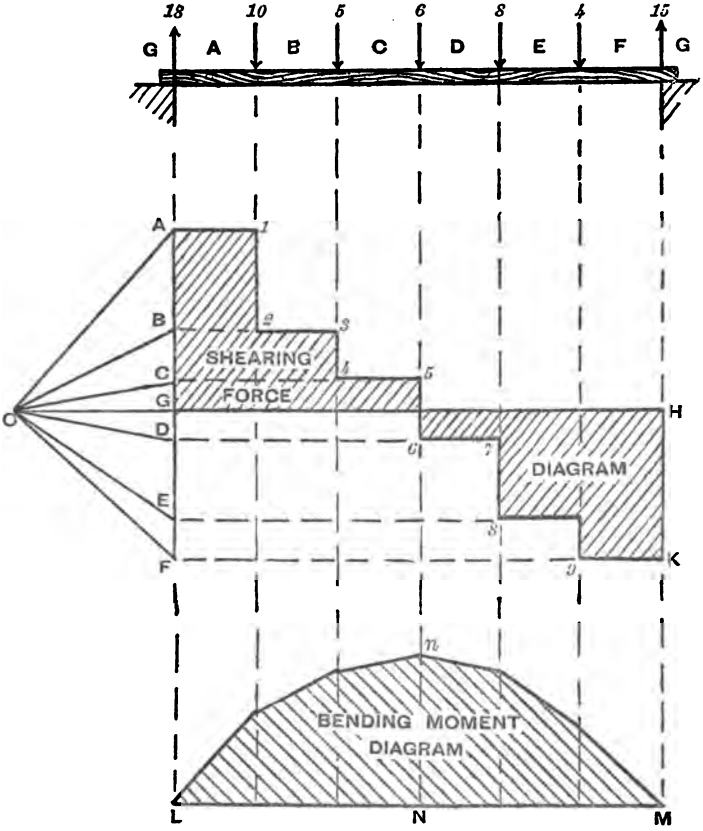
ANSWER.—The upper part of the figure shows the cantilever and the positions of the loads. Project down from these positions and the inner end of the beam, and then set out A B = 10 units, B C = 5, and C D = 5, to represent the forces A B, B C,



Cantilever in Example II.

and C D respectively. Draw horizontal lines through A, B, C, and D to intersect the lines of the forces. This gives us the Shearing Force Diagram as shown shaded.

To obtain the Bending Moment Diagram take any point O in E D, and join it to A, B, and C. Then take a base line M L parallel to O D, and draw M Q in the space C parallel to O C, Q P in the space B parallel to O B, and P N in the space A parallel to O A. Then L M Q P N is the Bending Moment Diagram, the scale being that adopted for the shear multiplied by the length of O D measured on the scale employed in setting out the length of the beam.



BEAM IN EXAMPLE III.

EXAMPLE III.—A beam of 12 feet span carries five loads equally spaced along its length, the first and last being each 2 feet from the nearest end. The values of the loads are 10, 5, 6, 8, and 4 tons respectively. Obtain graphically diagrams showing the shear and bending moment at every point of the beam.

ANSWER.—In this case we shall determine the reactions by calculation, thus:—

Reaction at left hand due to A B is $\frac{5}{6} \times 10 = 8\frac{1}{3}$ tons.
" " B C " $\frac{4}{6} \times 5 = 3\frac{1}{3}$ "
" " C D " $\frac{3}{6} \times 6 = 3$ "
" " D E " $\frac{2}{6} \times 8 = 2\frac{2}{3}$ "
" " E F " $\frac{1}{6} \times 4 = 0\frac{2}{3}$ "

$\therefore$ Total left-hand reaction G A = 18 "

The whole load is 33 tons, and therefore the right-hand reaction must be 33 – 18, or 15 tons.

We can now proceed as before, making G A = 18, A B = 10, B C = 5, &c., and drawing horizontal lines through A, B, C, &c., to obtain the Shearing Force Diagram.

Take a point O in the horizontal through G, and join it to A, B, C, &c.

Then the part of the Bending Moment Diagram in the space A is parallel to O A, in the space B to O B, in C to O C, and so on, as in Example II.

We might, of course, have determined the reactions from the Funicular Polygon L *n* M N in the first instance; but had we done so we would probably not have got the line O G horizontal, and would have had to redraw it as explained in the text.

Centre of Gravity of an Area.—Divide the area into elements, such as parallelograms, triangles, &c., the centres of gravity of which can be easily determined.

If the area is bounded by a curved line, divide it into very narrow strips, so that they may be considered approximately as parallelograms.

We have divided the area shown in Fig. 20 into three rectangles, and have found the centre of gravity of each. We first, assume a line lying in any direction, such as the line X X, along which the pull of gravity acts. The centre of gravity of each area is a point in the line of action of the pull of gravity on that area. The line of action of gravity will be parallel to this assumed line X X. The way may be towards either the left or the right as may be found most suitable, and the magnitude will be proportional to the area. The forces B C, C D, and D E represent completely the action of gravity on the top, the centre, and the bottom rectangles respectively.

Proceed to find the resultant of the three forces B C, C D, and D E, as explained for Fig. 1*a*. This is shown in Fig. 19 by the Polar Diagram B C D E O, and the corresponding funicular polygon 1 2 3 4. B C, C D, and D E, in the Polar Diagram are proportional to the areas of the three rectangles.

The line of action of the resultant of the three forces B C, C D, and D E passes through the centre of gravity of the whole area. This line is represented by the line 4—M. Now, assume a line at right angles to X X as a line along which the pull of gravity acts. Proceed in exactly the same way with regard to this line as has been done for the line X X, and we obtain another line passing through the centre of gravity of the whole area. The intersection M of these two resultants gives the centre of gravity of the whole area. The forces in the second case are called F G, G H, and H K, and the Polar Diagram F G H K O_1, with its corresponding Funicular Polygon, is shown in the figure.

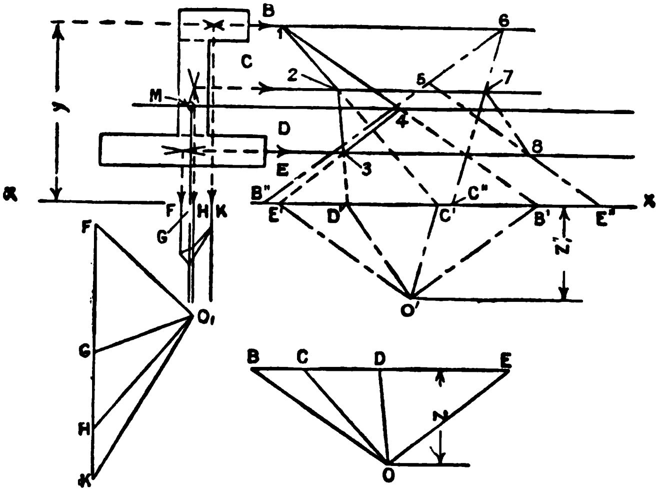


FIG. 20.—CENTRE OF GRAVITY AND MOMENT OF INERTIA.

Moment of Inertia of an Area.—If we wish to find approximately the Moment of Inertia round the line X X of the area in Fig. 20, we must first of all divide the area into elements just as in finding the centre of gravity. Then proceed to draw the polar diagram B C D E O and the corresponding funicular polygon 1 2 3 4

Now consider the top polygon and how we may determine its Moment of Inertia.

Produce the two lines derived from the polar diagram, which meet in the line of action of the pull of gravity on that area (viz., 1—2 and 1—4) until they intersect the line X X in the points B′, C′. Do the same for the lines 1—2 and 2—3, and 2—3 and 4—3, which meet on the lines of action of the pull of gravity on the middle and bottom areas. These lines intersect X X in the points C′, D′, and D′, E′, respectively.

Consider B′ C′, C′ D′, and D′ E′ as the magnitudes of the forces acting along the lines B C, C D, and D E respectively. Proceed as if to find their resultant by drawing the polar diagram B′ C′ D′ E′ O′ and the corresponding funicular polygon 5 6 7 8.

Produce, as before, the lines which meet in B C (viz., 6—5 and 6—7) to intersect the lines X X in the points B″ C″. Do the same for the lines 5—8 and 7—8, or, as we have done in the figure, produce the one which will cut X X in a point furthest from B″. B″ C″ measures to a certain scale the moment of inertia of the top area round the line X X, and B″ E″ the moment of inertia of the whole area round the same line. Greater accuracy would be obtained by dividing the area into smaller elements.

Proof.—Let y in Fig. 20 represent the distance the centre of gravity of the top area is from the line X X.

Now, since the two triangles B C O and B′ C′ 1 are similar:—

$$\mathrm{B\,C} : \mathrm{Z} :: \mathrm{B' \,C'} : y.$$

Then, $$\mathrm{B'\,C'} = \frac{\mathrm{B\,C} \times y}{\mathrm{Z}}.$$

Again, the two triangles B′ C′ O′ and B″ C″ 6 are similar:—

Hence, $$\mathrm{B'\,C'} : \mathrm{Z}_1 :: \mathrm{B''\,C''} : y.$$

And, $$\mathrm{B''\,C''} = \frac{\mathrm{B'\,C'} \times y}{\mathrm{Z}_1}.$$

Substituting the above value of B′ C′ we get:—

$$\mathrm{B''\,C''} = \frac{\mathrm{B\,C} \times y^2}{\mathrm{Z} \times \mathrm{Z}_1}.$$

But, B C × y^2 is the moment of inertia for the top area with respect to the line X X, provided the depth of the area is small in comparison with y. B″ C″ measured with the scale called the "area scale," as used for drawing B C (in order to represent the area of the top rectangle), gives the value of this moment of inertia, if Z and Z_1 are 1 unit of the scale which is used for setting off the lengths in drawing the section.

SCALE FOR MEASURING THE MOMENT OF INERTIA.—**Subdivide the unit of the scale used for representing the areas, into as many divisions as is represented by the number found by multiplying Z and Z_1, which are both measured by the length scale. One of these subdivisions will be the unit for the Moment of Inertia Scale. Or, measure $B'' E''$ with the area scale and multiply the reading first by Z and then by Z_1.**

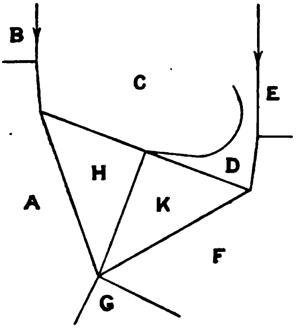


FIG. 21*a*.—FRAME DIAGRAM. ENGINE MECHANISM.

Engine Mechanism. — In the Frame Diagram, Fig. 21*a*, the bars B C and C E represent the centre lines of the piston-rods of a compound engine the heads of which are guided in parallel straight lines. The bars A C and D F are short connecting-rods, driving the crank F G, by means of the triangular frame shown. The joint C D K H is constrained to

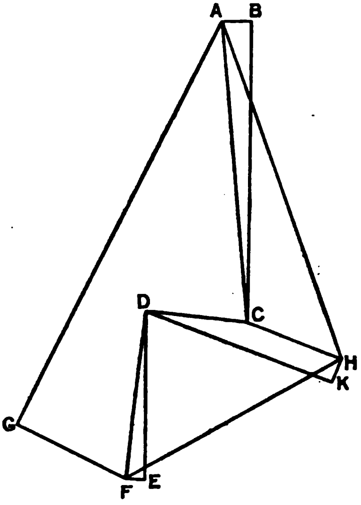
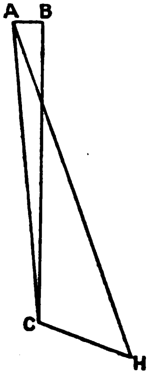


FIG. 21*b*.—STRESS DIAGRAM.



STRESS DIAGRAM FOR JOINT A C H.

move in an arc of a circle round a point in the bar C D produced towards the right. The bar C D is called a radius rod. The lines of action of all the external forces acting in the structure are shown. A B and E F are the guide pressures, C D the push or pull in the radius rod, F G the push or pull in the crank, and G A the crank effort or tangential resistance.

First Method.—We commence the Stress Diagram, Fig. 21*b*, by drawing D E to represent in magnitude the total pressure on the right-hand piston-rod. Then, E F and D F fix the point F, while F K and D K determine K.* But, we can get no further until we draw the Stress Diagram for the joint A C H. This is done by drawing B C to represent to the same scale as before the total pressure on the left-hand piston-rod. Then, the points A and H are determined.

We must now fit the Stress Diagram for the joint A C H, to the Stress Diagram already drawn; so that the point C shall lie on the line drawn through D parallel to the bar C D and the point H on the line drawn through K parallel to the bar K H, C H being kept parallel to the bar C H. Then the Stress Diagram, Fig. 21*b*, can be completed in the usual way.

Second Method.—Find the forces acting in A C and D F, and then find their resultant. Produce the line of action of this resultant to cut the line of action of the force C D; when, by joining this point with the crank pin, we get the line of action of the resultant force acting on the said crank pin. Finally, draw the Stress Diagram from the supplementary data.

The following is a list of books and papers on Graphic Statics and the Design of Structures:—

The Design of Structures, Bridges, Roofs, &c., by S. Anglin, C.E. (Chas. Griffin & Co., London, 1895.)

A Practical Treatise on Bridge Construction, by Prof. T. Claxton Fidler. (Chas. Griffin & Co., London.)

Graphical Determination of Forces in Engineering Structures, by James B. Chalmers, C.E. (Macmillan & Co., London.)

Graphic and Analytic Statics, by Robert Hudson Graham, C.E. (Crosby Lockwood & Co., London.)

Graphics, by Prof. R. H. Smith, M.Inst.M.E. (Longmans, Green & Co., London.)

Mechanics, vol. ii., by A. Jay Du Bois, C.E., Ph.D. (Chapman & Hall, London.)

Applied Mechanics, by Gaetano Lanza. (Chapman & Hall, London.)

Theory of Structures and Strength of Materials, by Henry T. Bovey, M.A., D.C.L. (Chapman & Hall, London.)

Graphic Methods of Computing Stresses in Jointed Structures. Paper by C. O. Burge, Proc. Inst. C.E. Vol. lxxiv., p. 192.

Graphic Methods of Engine Design, by A. H. Barker. (The Technical Publishing Co., Ltd., Manchester.)

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* The line F K has been omitted in the diagram.

Mechanical Graphics, by G. Halliday. (London, 1889.)

Elements of Graphic Statics, by K. von Ott, translated by G. S. Clark. (E. & F. N. Spon, London, 1888.)

Principles of Graphic Statics, by G. S. Clark. (E. & F. N. Spon, London, 1888.)

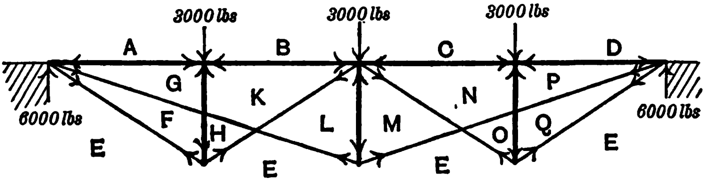
Elements of Graphic Statics, by L. M. Hoskins. (Macmillan & Co., London, 1892.)

Economics of Construction, by Robt. H. Bow. (E. & F. N. Spon, London.)

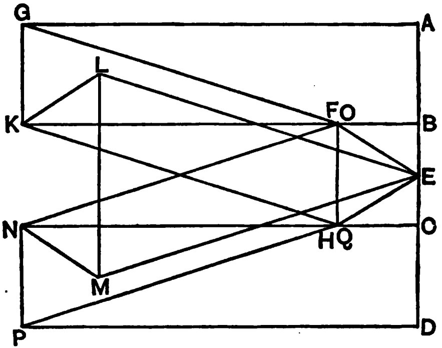
Applied Mechanics, 2nd edition, by Prof. James H. Cotterill. (Macmillan & Co., London.)

Lecture XXVIII.—Questions.

1. Draw the Stress Diagram for the Fink Truss shown below, and verify the stress diagram accompanying it.



FRAME DIAGRAM



STRESS DIAGRAM

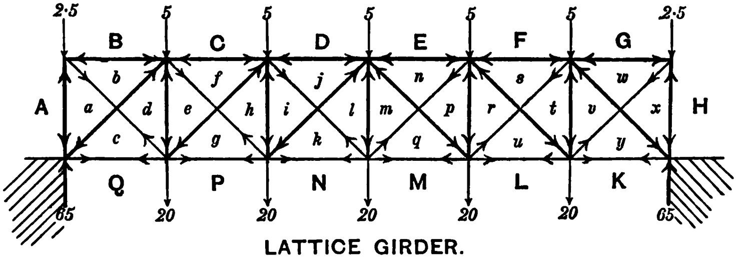
Fink Truss for Question 1.

2. A triangular frame is at rest under the action of three external forces. Prove that a certain diagram will represent the stresses in the bars of the frame. Extend this proposition to the case of a lattice girder of the Warren construction with four bays in the lower boom and three bays in the upper boom, loaded in the centre of the lower boom and supported at the ends, giving the Stress Diagram and showing how to distinguish the portions which are in compression or extension. (S. and A. Adv. Exam., 1889.)

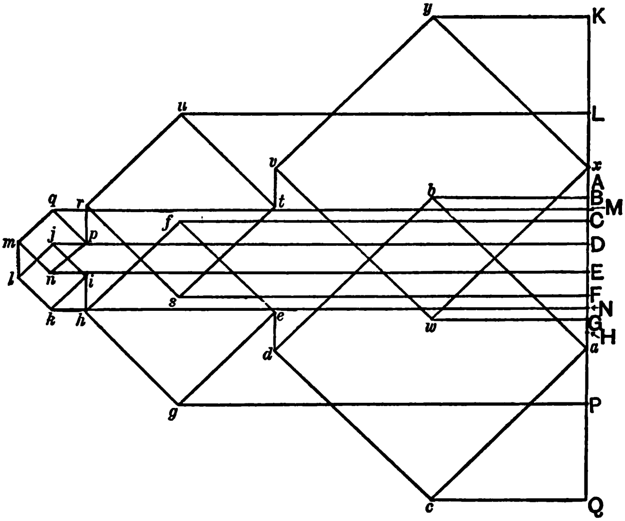
3. A triangular frame is acted on by three forces applied at its respective angular points and in equilibrium; investigate a method of constructing the diagram of all forces brought into play. Taking the case of a frame on the principle of the Warren Girder having four bays in the lower boom and three in the upper boom, and loaded at the centre of the lower member with a weight W, explain the method of constructing the diagram of forces,

drawing the same, and distinguishing those bars which act as struts from those which act as ties. (S. and A. Hons. Exam., 1892.)

4. The lower boom of a Warren Girder, supported at both ends, is divided into three bays. The upper boom has two bays, and the bracing bars are each inclined at 60° to the horizon. Find by graphic construction the stresses in the several pieces when the frame is loaded with 1,000 lbs. at the middle of the top boom.



LATTICE GIRDER.



STRESS DIAGRAM.

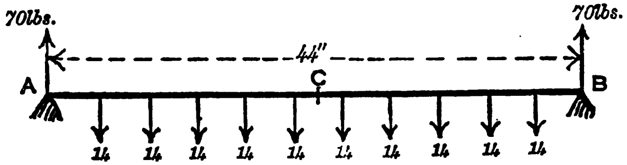
Lattice Girder for Question 7.

5. A Warren Girder has five bays consisting of equilateral triangles. If it be supported at each end and loaded at the two bottom central joints with loads of 18,000 lbs., find graphically the stress on each member, and show whether it is tensile or compressive. Explain fully the reasons and theory of the method you employ in obtaining your result. (S. and A. Adv. Exam., 1894.)

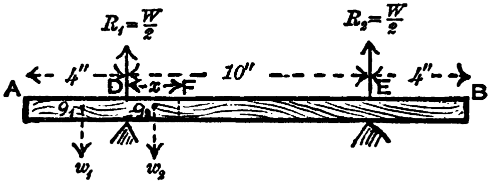
6. A Warren Girder of six bays with equilateral bracing, each bay being 10 feet long, is loaded with a distributed weight of 1 ton per foot run placed along the top of the girder; obtain the diagram of stress, and calculate the stresses in the various members, assuming that cotan. 60° = ·577, cosec. 60° = 1·155. *Ans.*

7. A lattice girder is loaded in the manner shown by the foregoing figure. Draw the Stress Diagram by the method explained in the text, and see if you get the same results as shown.

8. A bar of pine 44 inches long rests on props at its extremities, and just supports 10 weights, of 14 lbs. each, hung at equal intervals of 4 inches along the rod. Find graphically the B M at the centre of the bar and the amount of a single weight, which, if hung at the centre of the bar, would stress it to the same extent (see figure). *Ans.* 43·27 lbs.



Beam for Question 8.



Beam for Question 9.

9. A horizontal uniform bar, 18 inches long, is laid over two supports, each 4 inches from its ends, as shown in the figure. Find graphically two points at which the bending moment is zero, the bar being loaded by its own weight (see figure). *Ans.* 2 inches from inside of supports.

10. Given an iron arched rib, hinged at both ends, and a system of vertical loads, show how we find the stress at any point of any section. Prove the rule for stress at any point of a section when we know the resultant of all the forces acting on the structure on one side of the section.

PART V.—STRENGTH OF MATERIALS.

LECTURE XXIX.

Contents.—Stress—Definition of Intensity of Stress—Relation between Normal and Tangential Stresses—Strain—Example I.—Coefficient or Modulus of Elasticity—Limit of Elasticity—Work done in Stretching a Bar—Resilience—Example II.—Sudden Pull or Live Load—Shrunk Rings—Example III.—Strength of Thin Cylinders—Helical Seams—Strength of Thick Cylinders—Example IV.—Strength of Suspended Chains and Wires—Example V.—Questions.

Stress.—When a piece of material is subjected to the action of external forces they tend to cause the material to change its shape or form. The particular way in which the change takes place depends upon the manner in which the load is applied. This tendency gives rise to certain forces within the material which offer resistance to the change. These internal forces are generally called *stresses;* but the term *Stress* which we have now to consider has a somewhat more definite meaning. By the principle of the equality of action and reaction, we know that so long as no rupture of the material takes place, the algebraic sum of the components of the internal forces in the direction of the load at any section of the material must be equal to the load. This principle enables us to express the *internal* in terms of the *external* forces. It is a fundamental fact that, for a given load, the amount of resistance to be contributed by each individual fibre or part composing a section will be less or greater, according as the number of such fibres or parts is greater or less; or as we usually regard it, according as there is more or less area of section. This introduces us to the conception of *distributed* force, and paves the way towards gaining definite and clear ideas regarding the strength of materials.

Definition.—**Intensity of stress is the resistance or reaction due to a load per unit area of section.** For brevity it is usually called the Stress. Stresses may be of three different kinds, depending on the direction of the applied force with reference to the section on which the stress is estimated.

(1) If the applied force is normal or at right angles to the section, and acting *away* from it, the stress is called *tensile.*

(2) If acting *towards* the section, the stress is termed *compressive.*

(3) If the direction of the applied force be *parallel* to the section, then the stress is named a *shearing* stress.

It is evident that if the applied force be acting in a direction inclined to the given section, it will cause both a shearing and a direct stress, the latter being tensile or compressive, according as the force is directed away from or towards the section.

When the applied force acts in such a way that we know that its effect is uniformily distributed over the section we are considering, then we estimate the stresses as follows :—

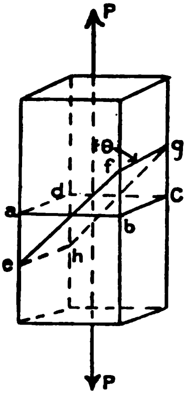
Let P_n = The applied load (or its component) acting normally to the section in lbs. or tons.

„ A = The area of the section (usually in square inches).

„ f = The direct stress, which may be either tensile or compressive.

„ P_t = The applied load (or its component) acting tangentially to the section in lbs. or tons.

„ f_s = The shearing stress.



ILLUSTRATING NORMAL AND TANGENTIAL STRESSES.

$$\left.\begin{aligned} &\text{Then,} \quad & f &= \frac{P_n}{A}. \\ &\text{And,} \quad & f_s &= \frac{P_t}{A}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(I)}$$

Relation between Normal and Tangential Stresses.—Let *a b c d* be the section of a bar normal to the direction of the applied force P, and *e f g h* another section making an angle θ, with the direction of P; and let the area of *a b c d* be A square inches.

Thus, the stress on *a b c d* is :—

$$f = \frac{P}{A};$$

But, on the area *e f g h*, we have a normal force :—

$$P_n = P \sin \theta,$$

And a tangential force :—

$$P_t = P \cos \theta.$$

Now, the area *e f g h* $= \dfrac{\text{area } abcd}{\sin \theta} = \dfrac{A}{\sin \theta}.$

If f_n and f_t be the normal and tangential stresses on the section *efgh*,

We have:— $$f_n = \frac{\frac{P_n}{A}}{\sin\theta} = \frac{P}{A} . \sin^2\theta = f . \sin^2\theta.$$

Similarly, we get:—$f = f . \sin\theta . \cos\theta.$

Strain.—When a piece of material, such as a bar of iron, is in tension or compression under the action of an applied force P, the bar will, in consequence, be lengthened or shortened by an amount depending on the extent to which it is *stressed.* The *ratio* which this *change* of length bears to the original length of the bar is called the *strain* due to P. Or in symbols,—

If, L = Original length of bar in inches.
And, l = Change of length of bar also in inches.

We have:— $$\text{Strain} = \frac{l}{L} \quad . \; . \; . \; . \; . \; . \; . \qquad \text{(II)}$$

Since L and l are both actual lengths, measured by some common unit, the student should carefully note that strain, as thus defined, is merely an *abstract ratio*, and *not* a quantity, for it is independent of the units employed.

EXAMPLE I.—A tie-rod, 100 ft. long, is stretched ¾ of an inch by the action of a certain force; what is the strain?

Here, $L = 100 \times 12 = 1{,}200$ inches,
And, $l = 0{\cdot}75$ inch.
$\therefore$ $$\text{Strain} = \frac{0{\cdot}75}{1{,}200} = 0{\cdot}000625.$$

Coefficient, or Modulus of Elasticity.—Experiment has demonstrated that for most materials used in engineering there is a very simple law connecting stress and strain, which is fairly well defined within certain limits. The stress is proportional to the strain, so long as the stress does not exceed a certain value, which, of course, is different for different materials and for different qualities of the same material. For example, if the stress be doubled, the strain will be doubled, or if the stress be reduced to one-half, the strain will also be halved, and so on. The limit beyond which this law does not hold is termed the **Limit of Elasticity.** When this limit is exceeded, the strain increases at a much greater rate than the stress producing it. Within the limit of elasticity, the material returns to its original state when the load is removed; but when stressed beyond this, the material does not do so, but retains a *permanent set.* In the following investigations the stress, in all cases, is assumed to be within the elastic limit:—

$$\text{Consequently,}\quad \frac{\text{Stress}}{\text{Strain}} = \text{E (a constant)}. \quad . \quad . \quad . \quad . \quad \text{(III)}$$

This constant E is termed the **Modulus of Elasticity**, or more appropriately by some writers the **Coefficient of Elasticity.**

Another way of exhibiting the relation subsisting among the various quantities we have been discussing is to combine equations (I), (II), and (III) in such a way as to express the stress and strain in terms of loads and dimensions.

$$\text{Thus,}\qquad E = \frac{P}{A} \div \frac{l}{L};$$

$$\text{Or,}\qquad P\,L = A\,l\,E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IV)}$$

Work done in Stretching a Bar.—Resilience.—If a load of *gradually* increasing amount be applied to a bar so as to stretch it, the amount of actual stretch, or elongation of the bar will, with the limitations already specified, be directly proportional to the load producing it. A diagram might, therefore, be drawn to represent graphically the work done in stretching the bar, as explained in Lecture II. of Volume I. The area of the diagram would represent the work done. The load will increase uniformly from 0 to P. The mean value of the force doing the work is, therefore, $\frac{1}{2}$ P, and the stretch or displacement is l. Hence, we have for the work done:—

$$W = \tfrac{1}{2}\,P\,l.$$

But from equations (I) and (III)—

$$P = fA, \qquad \text{and } l = \frac{fL}{E}.$$

$$\left.\begin{array}{ll} \text{Hence,} & W = \dfrac{f^2}{E} \times \dfrac{A\,L}{2} \\ \text{Or,} & W = \dfrac{f^2}{E} \times \tfrac{1}{2}\text{ volume of the bar.} \end{array}\right\} \quad . \quad . \quad . \quad . \quad \text{(V)}$$

The work done is therefore proportional to the volume of the bar, or to its weight.

When the bar is loaded to its elastic limit, or *proof stress*, as it is sometimes called, then the *work done* in stretching it is termed the **Resilience** of the bar, and the ratio $\frac{f^2}{E}$ is its **Modulus** or **Coefficient of Resilience.**

Example II.—What is the resilience of a material? If a wrought-iron tie bar, 5 feet long and 3 inches in diameter, has a limit of elasticity of 15 tons per square inch, and a modulus of elasticity of 30,000,000 lbs. per square inch, what is its resilience? (Take $\pi \doteq \frac{22}{7}$.) (Adv. S. & A. Exam. 1893).

ANSWER.—$f = 15 \times 2240$ lbs., $E = 30{,}000{,}000$ lbs. per square inch, $A = \frac{1}{4} \times \frac{22}{7} \times 3^2$ square inches, and $L = 5$ feet.

$$\therefore \text{ Resilience} = \frac{(15 \times 2240)^2}{30{,}000{,}000} \times \frac{\frac{11}{14} \times 3^2 \times 5}{2} = 665{\cdot}28 \text{ ft.-lbs.}$$

Sudden Pull, or Live Load.—We have just seen that a *constant* force of $\frac{1}{2}$P lbs. acting through a distance of l feet will do the same amount of work in stretching a bar as would a load gradually increasing from zero to P lbs.; therefore, the strain produced by a *sudden pull* of $\frac{1}{2}$P lbs. is the same as that due to P lbs. applied gradually. It follows, therefore, that if P be applied *suddenly*, but without initial velocity, the strain will be doubled, and the work done will be :—

$$W = P \times 2\,l = 2\,P\,l \text{ ft.-lbs.}$$

Or, in words, the work done on the bar by a suddenly applied or *live load* P, is *four* times that done by a gradually applied or *dead load* of the same amount.

Shrunk Rings.—In the construction of built-up guns, the process consists in shrinking on a series of concentric rings, each ring griping the next inner one with a certain pre-determined tension.

The reason for this arrangement will be better understood when we come to deal with the strength of thick cylinders. The principles set forth in the preceding sections enable us to calculate the dimensions of rings to give a certain grip.

Let D = The external diameter of an inner ring.
,, d = The internal diameter of the next outer ring.
,, f = The required tension.

When the outer ring is shrunk on, its diameter is then D. The inner fibres of this ring are then stretched by an amount $\pi\,(D-d)$; and by definition, we have :—

$$\text{Strain} = \frac{\pi\,(D-d)}{\pi\,d} = \frac{D-d}{d}.$$

If E denote the modulus of elasticity of the material of the ring, then :—

$$E = \frac{\text{stress}}{\text{strain}} = \frac{f}{\frac{D - d}{d}}$$

Hence, $$d = D\left(\frac{E}{E + f}\right).$$

Example III.—The external diameter of an inner ring is 20 inches. Work out the diameter which the outer ring must have in order to grip the inner one with an initial tension of 8 tons per sq. inch. Take the modulus of elasticity as 30,000,000.

Answer.—Here D = 20 inches, and $f = 8 \times 2240 = 17{,}920$ lbs. per sq. inch.

$$\therefore \quad d = 20 \times \frac{30{,}000{,}000}{30{,}017{,}920} = 19{\cdot}98 \text{ inches.}$$

Strength of Thin Cylinders.—By *thin* cylinders are meant cylindrical vessels whose thickness is small compared with their diameter. The resistance which such vessels offer to forces tending to burst them, both longitudinally and circumferentially, is easily deduced as follows :—Consider a cylindrical ring, whose breadth is b inches, thickness t inches, and internal diameter is d inches. Let p denote the intensity of the internal pressure, in lbs. per sq. inch, tending to burst the ring, and f the induced stress within the material of the ring, also in lbs. per sq. inch.

Then the magnitude of the total internal force tending to tear asunder the ring at the ends of a diameter is $p\,d\,b$ lbs. And the resistance which the ring offers to this bursting force is $2\,t\,b\,f$ lbs.

These being equal, we have :—

$$2\,t\,b\,f = p\,d\,b \quad \therefore \quad f = \frac{p\,d}{2\,t}. \quad . \quad . \quad . \quad \text{(VI)}$$

This result shows that the stress, in a circumferential direction, is independent of the length of the cylinder.

Whatever be the form of the ends of the cylinder—whether they be flat or hemispherical—the total force tending to cause rupture circumferentially is $p\,\frac{\pi}{4}\,d^2$ lbs. ; resisting this force, we have a ring of material whose total sectional area is $\pi\,d\,t$ sq. inches.

Let f_1 be the longitudinal stress due to the longitudinal bursting force; then the total resistance is $\pi\, d\, t\, f_1$ lbs.

And $$\pi\, d\, t f_1 = p \frac{\pi}{4} d^2$$

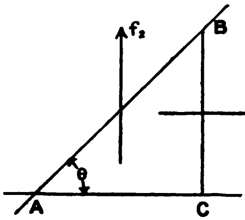
Hence, $$f_1 = \frac{p\, d}{4\, t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VII)}$$

From this we see that:—

$$f_1 = \tfrac{1}{2} f.$$

So that in a cylindrical boiler, which comes within the category of thin cylinders, the stress in a longitudinal direction is only one-half of the stress circumferentially.

Helical Seams.—If we made a boiler of rings, joined together circumferentially, then, so long as the strength of those joints was greater than one-half that of the solid plate, the boiler would still be as strong as one without joints, because the solid plate longitudinally would still be weaker than the circumferential joints. When, instead of solid rings, these are made up of pieces joined together longitudinally, it is obvious that the strength of the boiler is determined entirely by that of its longitudinal joints, unless the circumferential joints are less than half as strong.



ILLUSTRATING STRESS ON HELICAL SEAMS.

As a compromise, it has been proposed to have, instead of circumferential and longitudinal joints, one continuous seam running spirally, called a helical joint.

Let the accompanying figure represent a portion of such a boiler flattened out. A B is the helical seam, which, when flattened out, becomes a straight line, making the angle θ with the longitudinal direction. The longitudinal and circumferential stresses are represented by f_1 and f_2 respectively. The intensities of those stresses on A B being denoted by f_1' and f_2', we have:—

$$f_1' \times AB = f_1 \times BC; \qquad \text{and } f_2' \times AB = f_2 \times AC.$$

$$\therefore \qquad f_1' = f_1 \sin\theta; \qquad \text{and } f_2' = f_2 \cos\theta.$$

Resolving f_1' and f_2' normally to A B, we have, for the total normal stress:—

$$f_n = f_1' \sin\theta + f_2' \cos\theta$$

$$\text{,,} = f_1 \sin^2\theta + f_2 \cos^2\theta.$$

But, $$f_1 = \tfrac{1}{2} f_2.$$

∴ $$f_n = \tfrac{1}{2} f_2 \sin^2\theta + f_2 \cos^2\theta.$$

Or, $$\frac{f_n}{f_2} = 1 - \tfrac{1}{2}\sin^2\theta.$$

Let, $$n = \frac{\text{B C}}{\text{A C}};$$

Then, $$\sin^2\theta = \frac{\text{B C}^2}{\text{A B}^2} = \frac{\text{B C}^2}{\text{B C}^2 + \text{A C}^2} = \frac{n^2}{n^2+1}.$$

Hence, $$\frac{f_n}{f_2} = 1 - \tfrac{1}{2}\cdot\frac{n^2}{n^2+1} = \frac{n^2+2}{2n^2+2} \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII)}$$

For example, if $n = 1$, *i.e.*, $\theta = 45°$,

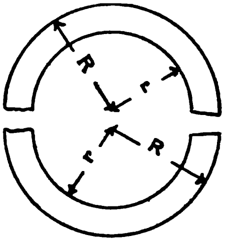
Then, $$\frac{f_n}{f_2} = \frac{3}{4}.$$

That is to say, that the normal stress on a spirally-running joint, making an angle of 45° with the axis of the boiler, would be three-fourths of that on a longitudinal joint. With joints of equal efficiency, therefore, the helical seam would be 33·3 per cent. stronger than the longitudinal one.

Strength of Thick Cylinders.—When the thickness of a cylindrical vessel, subjected to internal pressure, is *not* small in comparison with its internal diameter, the problem requires to be treated differently.

A complete determination of the strength of thick cylinders of all proportions is not an easy matter; and as for an *accurate* solution of the problem, the thing is simply impossible.

For moderate proportions of cylinders, such as are used in hydraulic appliances, the following demonstration yields results fairly substantiated by practice.



ILLUSTRATING STRAIN IN THICK CYLINDERS.

If such a cylinder were to give way under internal pressure, the plane of rupture would evidently contain the axis of the cylinder; whilst the rupture itself would appear as shown in the accompanying figure. From this figure it is clear that the circumferential *stretch* is the same from the inner to the outer surface. Now, remembering the definition of strain previously given, it is obvious that in this case, the strain in any cylindrical layer within the material will be inversely as its radius; and since, within the elastic limit, the stress is proportional to the strain, it follows that the stress on any layer is also inversely as the radius of that layer :—

Let f = Stress at inner surface at distance r, from axis of cylinder.
„ x = Radius of any layer within cylinder thickness.
„ dx = The thickness of elementary layer.

Then the stress on the material at radius x, will be $f\frac{r}{x}$, and the total resistance per unit length of the elementary hoop is $2fr.\frac{dx}{x}$.

The total resistance of the cylinder is, therefore :—

$$= 2fr\int_r^R \frac{dx}{x}$$

$$= 2fr \log_e \frac{R}{r}.$$

But the total bursting force within the cylinder, per unit of length, is $2pr$, where p is the difference of the internal and external pressures.

Hence, equating these expressions for equal and opposite forces, we have :—

$$2pr = 2fr \log_e \frac{R}{r}.$$

$$\frac{p}{f} = \log_e \frac{R}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IX)$$

The logarithms here required are hyperbolic.

Equation (IX) is not in a convenient form for application, because the ratio $\frac{R}{r}$ involves the quantity which is required; but a very simple and useful formula, quite accurate enough for most practical purposes, may be obtained as follows :—

For values of $\frac{R}{r}$ less than 2, $\log_e \frac{R}{r} = \frac{2(R-r)}{R+r}$, very approximately.

Making this substitution in (IX), we have :—

$$\frac{p}{f} = \frac{2(R-r)}{R+r}.$$

Whence, $$\frac{R}{r} = \frac{2f+p}{2f-p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (IXa)$$

In practical calculations, the quantity usually required is the thickness of the cylinder; calling this t, we get:—

$$t = R - r$$
$$= \left(\frac{2f+p}{2f-p}\right) r - r$$
$$= \frac{2pr}{2f-p}.$$

Or, $$t = \frac{pd}{2f-p}. \qquad \text{(X)}$$

Where d = internal diameter of cylinder.

Example IV.—The internal diameter of an hydraulic cylinder is 8 inches, and the ultimate tensile strength of the material of which it is made is 16,000 lbs. per sq. inch. What thickness of metal would be required in the sides of such a cylinder if the metal be not stressed beyond one-sixth of its ultimate strength, the water being under a pressure of 2000 lbs. per sq. inch? Prove the formula which you employ. (Hons. S. & A. Exam., 1889.)

Answer.—Here $f = \frac{1}{6} \times 16{,}000$; $p = 2000$; and $d = 8''$. Substituting these values in equation (X), we have:—

$$t = \frac{2000 \times 8}{2 \times \frac{1}{6} \times 16{,}000 - 2000} = \textbf{4·8 inches.}$$

If formula (IX) be used, then:—

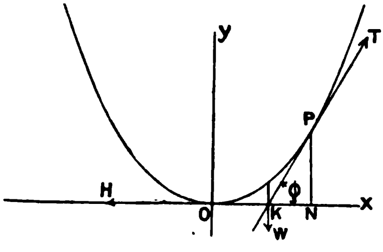
$$\log_e \frac{R}{r} = \frac{p}{f} = \frac{2000}{\frac{1}{6} \times 16{,}000} = 0·75.$$

Or, $$\log_{10} \frac{R}{r} = 0·75 \times ·4343 = ·326.$$

Therefore, $$\frac{R}{r} = 2·12.$$

And, $$t = R - r = \textbf{4·48 inches.}$$

Strength of Suspended Chains and Wires.—When a uniformly heavy chain or wire is suspended between two points, to find the equation to the curve in which it hangs, and the tension at any point. Let T be the tension at any point P; and H, that at the lowest point O. If W be the weight of the part O P of the chain, it is evident that O P will be in equilibrium under the action of three forces—the tensions at O and P and its own weight W, acting through its centre of gravity. These three forces, therefore, must pass through some point K, in O x, such that K P will be a tangent to the curve at the point P. Let the curve be referred to the co-ordinate axes, O x, O y, and let O N = x, and N P = y, O P = s. Also let w be the weight of a unit length of chain, and for H write $m\,w$. Resolving vertically and horizontally, we get:—



ILLUSTRATING STRENGTH OF SUSPENDED CHAINS.

$$\text{T} \sin \varphi = \text{W} = s\,w. \quad \text{T} \cos \varphi = \text{H} = m\,w.$$

Hence, $\tan \varphi = \frac{s}{m}$; or, $\frac{dy}{dx} = \frac{s}{m}$. . . . . (1)

But, $$\frac{dy}{ds} = \sin \varphi = \frac{1}{\operatorname{cosec} \varphi} = \frac{1}{\sqrt{\cot^2 \varphi + 1}} = \frac{s}{\sqrt{m^2 + s^2}}$$

$$\therefore \quad dy = \frac{s \, . \, ds}{\sqrt{m^2 + s^2}}.$$

Integrating this expression, we have:—

$$y = \sqrt{m^2 + s^2} + \text{C}.$$

To find the value of the constant C, we know that $s = 0$, when $y = 0$.

$$\therefore \quad 0 = m + \text{C},$$

Or, $$\text{C} = -m.$$

So that, $$y + m = \sqrt{m^2 + s^2}.$$

and, therefore, $$s = \sqrt{(y + m)^2 - m^2}.$$

Substituting this value of s in (1), and inverting:—

$$\frac{dx}{dy} = \frac{m}{\sqrt{(y + m)^2 - m^2}}.$$

Multiplying each side by dy, and integrating, we get:—

$$x = m \, . \, \log_e \{(y + m) + \sqrt{(y + m)^2 - m^2}\} + \text{C}.$$

When $x = 0$, $y = 0$, then:—$\text{C} = -m \, . \, \log_e m$.

Hence, $$x = m \, . \, \log_e \left\{ \frac{y + m) + \sqrt{(y + m)^2 - m^2}}{m} \right\} \quad . \quad . \quad (2)$$

Equation (2) is sometimes useful in the solution of problems. In order to get the equation to the curve in which the chain hangs, or, in other words, the relation between x and y, we write (2) as follows :—

$$\frac{x}{m}=\log_e\left\{\left(\frac{y+m}{m}\right)+\sqrt{\left(\frac{y+m}{m}\right)^2-1}\right\}$$

That is, $$e^{\frac{x}{m}}=\left(\frac{y+m}{m}\right)+\sqrt{\left(\frac{y+m}{m}\right)^2-1}\,; \quad . \quad . \quad . \quad . \quad (3)$$

Where e is the base of the Napierian or hyperbolic logarithms.

Now, since :—

$$\left\{\left(\frac{y+m}{m}\right)+\sqrt{\left(\frac{y+m}{m}\right)^2-1}\right\}\cdot\left\{\left(\frac{y+m}{m}\right)-\sqrt{\left(\frac{y+m}{m}\right)^2-1}\right\}=1$$

$$\therefore\left\{\left(\frac{y+m}{m}\right)-\sqrt{\left(\frac{y+m}{m}\right)^2-1}\right\}=\frac{1}{\left\{\left(\frac{y+m}{m}\right)+\sqrt{\left(\frac{y+m}{m}\right)^2-1}\right\}}$$

$$=\frac{1}{e^{\frac{x}{m}}}.$$

Hence, $$e^{-\frac{x}{m}}=\left(\frac{y+m}{m}\right)-\sqrt{\left(\frac{y+m}{m}\right)^2-1}. \quad . \quad . \quad . \quad . \quad (4)$$

Now, adding (3) and (4) and reducing, we have finally :—

$$\left.\begin{aligned} y&=\frac{m}{2}\left\{e^{\frac{x}{m}}+e^{-\frac{x}{m}}-2\right\}\\ \text{Or,}\quad y&=\frac{m}{2}\left\{e^{\frac{x}{2m}}-e^{-\frac{x}{2m}}\right\}^2\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (\text{XI})$$

The curve whose equation is (XI) is called a *catenary*.

To find T, the tension at any point, we have :—

$$\mathrm{T}=\frac{s\,w}{\sin\phi}=s\,w\div\frac{s}{\sqrt{s^2+m^2}}$$
$$=w\,.\,\sqrt{s^2+m^2}.$$

But $$\sqrt{s^2+m^2}=y+m.$$

$$\therefore \qquad \mathrm{T}=w\,(y+m). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XII})$$

When the curve is very flat, as in the case of telegraph wires, then $s=x$ approximately, and (1) becomes $\frac{dy}{dx}=\frac{x}{m}$.

Multiplying by $d\,x$ and integrating, we get:—

$$y = \frac{x^2}{2\,m}. \quad \ldots\ldots\ldots \quad \text{(XIII)}$$

This requires no correction, because x and y vanish together.

Example V.—A telegraph wire, which weighs $\frac{1}{10}$ of a lb. per yard, is stretched between poles on level ground, so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 lbs. (Hons. S. and A. Exam., 1891.)

Answer.—Here, $H = 140$, $y = 3$ ft., and $w = \frac{1}{30}$; and since:—

$$H = m\,w$$

$$\therefore \quad 140 = m \times \frac{1}{30}.$$

Or, $m = 4200.$

Putting these values of y and m in equation (XIII), it becomes:—

$$3 = \frac{x^2}{2 \times 4200}.$$

$$\therefore \quad x = \sqrt{3 \times 2 \times 4200} = 158{\cdot}7 \text{ ft.}$$

Distance between poles:—

$$= 2\,x = 2 \times 158{\cdot}7 = \mathbf{317{\cdot}4\ ft.}$$

If the more exact equation (2) be used, then:—

$$2\,x = 2 \times 4200 \times \log_e \left\{ \frac{4203 + \sqrt{4203^2 - 4200^2}}{4200} \right\}$$

$$,, = 8400 \times \log_e \frac{218}{210}$$

$$,, = 8400 \times 0{\cdot}0374 = \mathbf{314{\cdot}16\ ft.}$$

LECTURE XXIX.—QUESTIONS.

1. What do you understand by the terms *strain, stress*, and *modulus of elasticity?* A tie rod 100 feet long, and of 2 square inches sectional area, is stretched three-quarters of an inch under a tension of 32,000 lbs. What is the intensity of the stress, the strain, and the modulus of elasticity under these circumstances? (S. and A. Exam., 1888.) *Ans.* 16,000 lbs. per square inch; 0·000625; 25,600,000.

2. A ship is moored by two cables of 90 feet and 100 feet in length respectively. The first cable stretches 2$\frac{5}{8}$ inches, and the second stretches 3 inches, under the pull of the ship; find the strain of each cable. (S. and A. Exam., 1889.) *Ans.* 0·00243; 0·0025.

3. Define the term *Resilience.* Show that the work done on a material by a *live load* is four times that done by an equal *dead load.* A wrought-iron tie rod 20 feet long and ·5 square inch cross sectional area bears a dead load of 5,000 lbs. Find the work done on stretching the rod by this load. What live load would produce an instantaneous elongation of another $\frac{1}{10}$ inch? Take E = 30,000,000. *Ans.* 33·$\dot{3}$ ft.-lbs.; 3,125 lbs.

4. A rod of iron 25 feet long and 2 square inches cross sectional area checks a weight of 80 lbs., which falls from a height of 20 feet before beginning to strain it. Find the greatest stress and strain produced. Take E = 25,000,000. *Ans.* 39,960 lbs. per square inch; ·0016.

5. If the modulus of elasticity of a piece of steel in lbs. per square inch is 32,000,000, how much would a bar $\frac{3}{4}$ of an inch in diameter and 25 inches long extend under a load of 10 tons? If its limit of elasticity is 21 tons per square inch, what is its resilience? (S. and A. Exam., 1894.)

6. What is the resilience of a bar? A bar of steel is $\frac{7}{8}$ inch in diameter, and 30 inches in length, and is under a tensile pull of 10 tons, what is the work stored up in the bar, the modulus of elasticity being 32,000,000 lbs. per square inch? (S. and A. Exam., 1895.)

7. Built-up guns are made of concentric rings, the outer hoops, or rings, being shrunk or forced upon inner tubes with a regulated tension. Supposing the external diameter of the inner tube to be 12 inches, and that the substance of its covering hoop is to have given to it an initial grip of 4 tons per square inch of its sectional area; the exterior diameter of this second hoop is 18 inches, and is to be covered with a third hoop, having an initial grip of 8 tons per square inch of its sectional area; will you work out in arithmetic the difference of dimensions that will afford the above conditions?

8. Prove that when a thin spherical shell is exposed to the bursting pressure of gas or liquid the stress in the material is half as great as that within the curved surface of a thin cylindrical shell exposed to the like pressure, each shell being of the same thickness and diameter. (S. and A. Exam., 1891.)

9. A long thin pipe of given internal radius is subjected to fluid pressure; find the tension of the material of the pipe. If the internal radius of the pipe is 6 inches, and the thickness of the pipe 0·5 inch, what fluid pressure per square inch would increase the radius of the pipe by 0·001 inch? The modulus of elasticity being 20,000,000, and the elasticity of the material being supposed to continue perfect. *Ans.* 277·7 lbs. per square inch.

10. A *steel* hydraulic cylinder, 10 feet long and 6 inches in diameter, acts as a brake on a lift. It has a movable piston fitted with a spring valve, the cylinder being full of liquid when the lift is at its highest position, and the piston and rod at the end of the stroke inside the cylinder. It was found that when the lift began to descend the internal pressure was 1,000 lbs. per square inch, which gradually rose to 2,000 lbs. when the piston had travelled 9 feet. Treating the cylinder as a thin one, what would be the law of variation of thickness at different points? Prove the formula. (S. and A. Hons. Exam., 1890.)

11. A uniformly heavy chain is suspended from two given points: find the equation to the curve in which it hangs, and the tension at any point of the curve. (S. and A. Hons. Exam., 1892.)

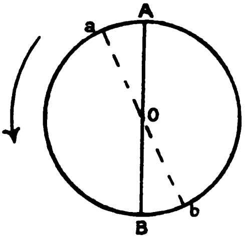
LECTURE XXX.

CONTENTS.—Torsional Strength of Shafts—Examples I., II., and III.—Strength of Shafts subjected to Combined Twisting and Bending—Theorem—Examples IV. and V.—Stiffness of Shafts—Angle of Twist—Example VI.—Questions.

Torsional Strength of Shafts.—In order to transmit energy through a shaft, the driving force must be applied at some distance from its centre. The driving force and its effective leverage, therefore, constitute what is termed a **Turning** or **Twisting Moment** (T.M.) which puts the shaft in a state of twist or torsion. The tendency of a purely torsional moment applied to a shaft is to cause the shaft to shear in planes normal to its axis, and this has to be met by the shearing resistance of the material, which resistance must, of course, be of the nature of a moment. The resistance the shaft offers to twisting we term its **Torsional Resistance** (T.R.); and as this balances the turning moment, we have :—

$$\text{T.M.} = \text{T.R.}$$

We have now to find the value of T.R., as depending on the material and dimensions of the shaft, and shall confine ourselves to shafts of circular section—solid and hollow. Suppose the accompanying figure to represent an end view of a shaft; and suppose A B and *a b* to have been parallel diameters of two sections very near to each other when the shaft was at rest; then, when the shaft is at work transmitting energy, the diameters, A B and *a b*, will no longer be parallel, but will make an angle with each other, as shown. A longitudinal section, through the axis of a shaft, which is a plane when the shaft is at rest, thus becomes a screw surface when the shaft is working. We shall have occasion later to measure this angle of twist; but in the meantime we are mainly concerned with the distribution of shearing stress within the shaft.



ILLUSTRATING STRAIN IN A SHAFT.

Looking at the figure, we easily see that the *strain* in any ring of fibres must be proportional to the arc of this ring which is included between the diameters A B and $a\,b$, when these are twisted out of parallelism by the turning moment. Within the elastic limit of the material, therefore, it follows that the shearing *stress* in any ring of fibres is proportional to the radius of that ring.

Therefore, let f = the greatest shearing stress, in lbs. per sq. inch, permissible in the material of the shaft.
D = the outside diameter,
d = the inside diameter of the shaft, both in inches.
And x = the radius of *any* ring of fibres within the material of the shaft.

Then the shaft must be so proportioned that f shall be the value of the stress in its outermost fibres which are $\frac{1}{2}$ D inches from the centre. Consequently, from what has already been said, we have :—

$$\text{Stress at } x = \frac{x}{\frac{1}{2}D} f = \frac{2\,x}{D} f.$$

Consider, now, the ring of fibres at x inches from the shaft centre, whose radial thickness is $d\,x$ inches. The sectional area of this elementary ring will $= 2\,\pi\,x\,d\,x$ sq. inches; and its resistance to shearing will be

$$2\,\pi\,x\,d\,x \times \frac{2\,x}{D} f \text{ lbs.} = \frac{4\,\pi f}{D} x^2\,d\,x \text{ lbs.}$$

Now, the leverage at which this resisting ring of fibres acts, is x inches; therefore, its *moment* of resistance is $\frac{4\,\pi f}{D} x^2\,d\,x \times x$, or $\frac{4\,\pi f}{D} x^3\,d\,x$ inch-lbs.

Hence, summing up the moments of resistance of all such elementary rings which go to make up the shaft, we get:—

$$\text{T.R.} = \frac{4\,\pi f}{D} \int_{\frac{1}{2}d}^{\frac{1}{2}D} x^3\,d\,x,$$

$$\text{,,} \quad = \frac{4\,\pi f}{D} \left\{ \frac{(\frac{1}{2}D)^4}{4} - \frac{(\frac{1}{2}d)^4}{4} \right\}$$

$$\text{,,} \quad = \frac{\pi}{16} f \left(\frac{D^4 - d^4}{D} \right).$$

Hence, for hollow shafts, we have:—

$$\text{T.R.} = \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)f \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I})$$

For solid shafts, we make $d = 0$, and get:—

$$\text{T.R.} = \frac{\pi}{16}D^3 f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II})$$

It is instructive to compare the torsional resistances of solid and hollow shafts of the same weight and material. For this purpose let D_1 be the outer diameter of hollow shaft.

Then, if we neglect couplings, and consider the shafts to be of equal length, the weights will simply be proportional to their sectional areas; *i.e.*:—

$$\frac{\text{Weight of hollow shaft}}{\text{Weight of solid shaft}} = \frac{D_1^2 - d^2}{D^2}.$$

For equal weights, this ratio is unity; therefore we have the relation:—

$$D^2 = D_1^2 - d^2$$

Or, $$D = \sqrt{D_1^2 - d_1^2}.$$

Now, we have from equations (I) and (II):—

$$\frac{\text{T.R. of hollow shaft}}{\text{T.R. of solid shaft}} = \frac{D_1^4 - d^4}{D_1 \times D^3} = \frac{D_1^2 + d^2}{D_1 \times D} \times \frac{D_1^2 - d^2}{D^2}$$

$$\text{"} \qquad \text{"} \qquad = \frac{D_1^2 + d^2}{D_1 \times D} = \frac{D_1^2 + d^2}{D_1 \times \sqrt{D_1^2 - d^2}}.$$

It will simplify matters if we put $d = x \times D_1$, where x is a proper fraction, we then have:—

$$\frac{\text{T.R. of hollow shaft}}{\text{T.R. of solid shaft}} = \frac{1 + x^2}{\sqrt{1 - x^2}}.$$

For example, let $x = \frac{1}{2}$, then:—

$$\frac{1 + x^2}{\sqrt{1 - x^2}} = \frac{1 + \frac{1}{4}}{\sqrt{1 - \frac{1}{4}}} = \frac{5}{2\sqrt{3}} = 1{\cdot}443.$$

This result shows that for the same length and weight, the hollow shaft having outer and inner diameters in the proportion of 2 to 1 will be 44·3 per cent. stronger than the solid one.

The turning moment driving a shaft may either be uniform or variable in amount. Shafts driven by means of gearing, and revolving at a uniform speed, are generally considered as cases of uniform turning moment. As a typical example of variable turning moment, we have the case of the steam engine crankshaft, where both the driving force of the steam on the piston and its effective leverage are continually varying throughout the stroke.

When the turning moment is uniform—that is, when the shaft revolves uniformly at n revolutions per minute, and transmits energy at the rate of so many H.P., this is all the data we require to know in order to estimate T. M. We have already seen (see Vol. I., Lect. III.) that the work done by a turning couple in one minute is equal to the magnitude of the turning couple multiplied by its angular displacement in the same time. Now our turning couple, or turning moment, as we call it, is T.M. inch-lbs., or $\frac{1}{12}$ T.M. foot-lbs., and the angular velocity of our shaft is $n \times 2\pi$ radians per minute.

Therefore, the

$$\text{Work done} = \frac{\text{T.M.}}{12} \times 2\pi n \text{ ft.-lbs. per minute}$$

And the

$$\text{H.P.} = \frac{\frac{\text{T.M.}}{12} \times 2\pi n}{33{,}000} = \frac{n \times \text{T.M.}}{63{,}024}.$$

$$\therefore \quad \text{T.M.} = 63{,}024 \,.\, \frac{\text{H.P.}}{n}. \quad \ldots\ldots \quad \text{(III)}$$

Example I.—Find the moment of resistance to torsion of a hollow shaft. Compare the strengths to resist torsion of a solid and hollow shaft of the same length and weight, the extreme diameter of the hollow shaft being double its internal diameter. A hollow shaft, the external and internal diameters of which are 20 inches and 8 inches respectively, runs at 70 revolutions per minute, with a surface stress of 6,000 lbs. per square inch; find the twisting moment and the horse-power transmitted. (S. & A. Hons. Exam., 1895.)

Answer.—The first two parts of this question have already been answered in the text.

With regard to the last part, we are asked to find the values of T.M. and H.P., being given:—

D_1 = 20 inches.	f = 6000 lbs. per sq. in.
d = 8 inches.	n = 70 per min.

Since T.R. = T.M.

$$,, = \frac{\pi}{16} \cdot \frac{D_1^4 - d^4}{D_1} \cdot f.$$

$$\therefore \qquad \text{T.M.} = \frac{3{\cdot}1416}{16} \times \frac{20^4 - 8^4}{20} \times 6000.$$

$$,, \quad = \mathbf{9{,}183{,}525 \text{ inch-lbs.}}$$

$$\text{and} \qquad \text{H.P.} = \frac{\text{T.M.} \times n}{63{,}024}$$

$$,, \quad = \frac{9{,}183{,}525 \times 70}{63{,}024}$$

$$,, \quad = \mathbf{10{,}200.}$$

Example II.—If a steel shaft revolving at 60 revolutions per minute be required to transmit 220 horse-power, what should be its diameter so that the maximum stress produced in it may not exceed one-fifth of that at the elastic limit? The elastic limit in torsion is 18 tons per sq. inch. Prove any formula you may employ. (S. & A. Hons. Exam., 1894.)

Answer.—Combining formulæ (II) and (III) we have:—

$$\text{T.R.} = \text{T.M.},$$

$$i.e., \qquad \frac{\pi}{16} D^3 f = 63{,}024 \times \frac{\text{H.P.}}{n}.$$

$$\therefore \qquad D = 68{\cdot}5 \sqrt[3]{\frac{\text{H.P.}}{n f}}. \quad . \quad . \quad . \quad \text{(IV)}$$

Here, H.P. = 220. $n = 60$.

And, $f = \frac{1}{5} \times 18 \times 2240 = 8064$ lbs. per sq. in.

$$\therefore \qquad D = 68{\cdot}5 \times \sqrt[3]{\frac{220}{60 \times 8064}} = \mathbf{5{\cdot}27 \text{ inches.}}$$

In cases where the turning moment exerted on a shaft varies, it is, of course, necessary that the shaft should be of strength sufficient to withstand safely the maximum value of T.M. So that in dealing with an example like that of the steam engine crank-shaft we take as the turning force the product of the maximum effective steam pressure on the piston into the piston area; and for the leverage we take the crank radius, although this is not quite accurate; because, if the crank be driven by means of a connecting-rod, the virtual leverage of the steam force at a certain point in the stroke exceeds that of the crank radius by an amount depending on the relative lengths of the crank and connecting-rod.

But on the other hand, the effective steam pressure on the piston is, as a rule, much below its maximum value when the piston reaches the point of greatest leverage. On the whole, therefore, it is quite accurate enough for all practical purposes to estimate the maximum turning moment in the way we have indicated.

Thus, Let p = Greatest effective steam pressure acting on the piston, in lbs. per sq. inch.

„ A = Area of piston, in sq. inches.

„ r = Crank-radius, in inches.

Then, max. T.M. = $p\,A\,r$ inch-lbs.

By *effective* steam pressure, we mean the *difference* between the pressures behind, and in front of, the piston.

EXAMPLE III.—Find the diameter of the crank-shaft for a horizontal engine which is to be worked with an effective mean steam pressure of 45 lbs. per square inch throughout the stroke, the diameter of the cylinder being 36 inches, the stroke 5 feet, and the working load being taken at $\frac{1}{6}$ of the breaking load. The shaft is to be of wrought iron, such that a 1-inch shaft will break with the torsion produced by 800 lbs. acting at the end of a 12-inch lever. (S. & A. Hons. Exam.)

ANSWER.—Let f_B be the breaking stress of the experimental shaft, then the working stress in the crank shaft, according to the question, will be $\frac{1}{6}f_B$.

To find the value of f_B we are given that when T.M. = 800 × 12 inch-lbs., and D = 1″, fracture takes place. From these data, therefore, we deduce :—

$$f_B = \frac{800 \times 12}{\frac{\pi}{16} \times 1^3} = \frac{800 \times 12}{\frac{\pi}{16}} \text{ lbs.}$$

The area of a 36-inch piston = 1017·87 square inches, and r is 30 inches.

$$\therefore \quad \text{Max. T.M.} = 45 \times 1017{\cdot}87 \times 30 \text{ inch-lbs.}$$

$$\text{Also,} \quad ,, \quad = \frac{\pi}{16} D^3 f.$$

$$\therefore \quad D^3 = \frac{45 \times 1017{\cdot}87 \times 30}{\frac{\pi}{16} f};$$

but,
$$f = \frac{1}{6} f_{B} = \frac{800 \times 2}{\frac{\pi}{16}}.$$

Hence,
$$D = \sqrt[3]{\frac{45 \times 1017{\cdot}87 \times 30}{800 \times 2}},$$

$$,, = 9{\cdot}5 \text{ inches, nearly.}$$

Strength of Shafts subjected to combined Twisting and Bending.—In Example III. the diameter of the shaft has been calculated as for a purely twisting moment. But in no case of a shaft being driven by a crank is the effect of the load quite so simple as this. Besides the turning moment, which we have already seen how to deal with, there is always in action a *bending* moment of greater or less magnitude depending on the engine arrangement. The worst case is that in which the crank is overhung. When this is so, the bending moment is caused by the load on the piston acting along a line (the centre line of the cylinder) at a certain distance from the shaft bearing nearest to the crank.

Let l = the distance between the centre line of the cylinder and the middle of the nearest shaft bearing, in inches; and

p and A = (as before) the effective steam pressure and piston area respectively.

Then the magnitude of the bending moment which we have now to take into account is

$$\text{B.M.} = p\,\text{A}\,l \text{ inch-lbs.}$$

This bending moment is balanced by the moment of resistance of the shaft, which, as will be shown in the next lecture, is

$$\text{M.R.} = \frac{\pi}{32}\,\text{D}^3 f_t\,;$$

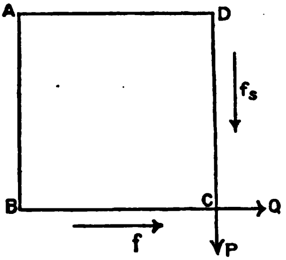
Where, D = diameter of the shaft journal, in inches,

And, f_t = the tensile stress in the outer fibres of the journal, in lbs. per sq. inch.

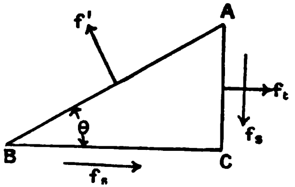
Hence, we see that when a crank-shaft is being turned by the steam on the piston, it is subjected simultaneously to a shearing stress of intensity f_s, and a tensile stress of intensity f_t. The problem now before us is to combine these stresses so as to obtain what is termed the *Equivalent* tensile or shearing stress; but in order to render all the steps in the process clear and intelligible, we require to demonstrate the following theorem :—

Theorem.—A shearing stress on any plane produces a shearing stress of equal intensity on planes at right angles to it.

Let A B C D be a rectangular block of material whose thickness is 1 inch perpendicular to the plane of the paper. And let f_s be the intensity of the shearing stress over the face whose edge is C D. It is easy to see that the total shearing force on the face C D which tends to pull that face parallel to itself, must be accompanied by a similar effect on the face B C in order that the block may not be turned around A. To find the relation between those forces, take moments about A, and we get :—



ILLUSTRATING SHEARING STRESS THEOREM.



ILLUSTRATING EQUIVALENT TENSILE STRESS.

$$P \times A\,D = Q \times A\,B.$$

Or, $$(f_s \,.\, C\,D) \times B\,C = (f \,.\, B\,C) \times C\,D.$$

$$\therefore \quad f_s = f.$$

Hence, we see that the shearing stress induced in a shaft by the turning moment is accompanied by a shearing stress of equal intensity on planes at right angles to it; that is, parallel to the axis of the shaft.

In the right-hand figure let A C represent the edge of a small portion of a plane normal to the axis of the shaft, and B C that of another plane at right angles to A C. On the former of these planes there is a shearing stress of intensity f_s, due to the turning moment, and a direct tensile stress of intensity f_t, due to the bending moment acting on the shaft. By the theorem just proved, we also have on B C a shearing stress f_s. Let f' denote the intensity of a tensile stress, which, acting on a plane A B inclined to A C and B C, would balance the stresses on these latter planes. As before, let the width of the three planes perpendicular to the plane of the paper be unity.

Resolving vertically and horizontally, we have :—

$$(f' \,.\, \mathrm{A\,B}) \cos\theta = (f_s \,.\, \mathrm{A\,C}),$$

and $$(f' \,.\, \mathrm{A\,B}) \sin\theta = (f_s \,.\, \mathrm{B\,C}) + (f_t \,.\, \mathrm{A\,C}).$$

From the first of these equations we get :—

$$\frac{f'}{f_s} = \frac{\frac{\mathrm{A\,C}}{\mathrm{A\,B}}}{\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta, \quad . \quad . \quad . \quad . \quad (1)$$

and from the second :—

$$f' = \frac{\frac{\mathrm{B\,C}}{\mathrm{A\,B}}}{\sin\theta} . f_s + \frac{\frac{\mathrm{A\,C}}{\mathrm{A\,B}}}{\sin\theta} . f$$

$$,, = \frac{\cos\theta}{\sin\theta} f_s + \frac{\sin\theta}{\sin\theta} f_t.$$

$$,, = f_s \cot\theta + f_t.$$

Or, $$\frac{f'-f_t}{f_s} = \cot\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying together (1) and (2), we get :—

$$\frac{f'}{f_s} . \frac{f'-f}{f_s} = 1.$$

$$\therefore \qquad f'(f'-f) = f_s^2.$$

Which on being solved for f' gives :—

$$f' = \frac{f_t}{2} + \sqrt{\frac{f_t^2}{4} + f_s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{V})$$

We take the positive sign in the solution of this quadratic equation, for obviously f' is greater than $\frac{1}{2} f_t$.

Being now in possession of the relation subsisting among the stresses, we next have to express these in terms of the T.M. and B.M. :—

Since $$\text{T.M.} = \frac{\pi}{16}\,D^3 f_s.$$

And $$\text{B.M.} = \text{R.M.} = \frac{\pi}{32}\,D^3 f_t.$$

Hence, $$f_s = \frac{\text{T.M.}}{\frac{\pi}{16}\,D^3} \quad \ldots\ldots\ldots \quad (3)$$

And $$f = \frac{\text{B.M.}}{\frac{\pi}{32}\,D^3} \quad \ldots\ldots\ldots \quad (4)$$

In like manner, we must have:—

$$f' = \frac{\text{B.M.}'}{\frac{\pi}{32}\,D^3}$$

where B.M.′ stands for the *equivalent* bending moment.

Making these substitutions, and reducing, (V) becomes:—

$$\text{B.M.}' = \tfrac{1}{2}\,\{\text{B.M.} + \sqrt{\text{B.M.}^2 + \text{T.M.}^2}\} \quad \ldots \quad (\text{VI})$$

Now, if T.M.′ denotes the *equivalent* twisting moment, it easily follows from equations (3) and (4) that for equal intensities of stress we have:—

$$\text{T.M.}' = 2\,\text{B.M.}'$$

Hence, $$\text{T.M.}' = \text{B.M.} + \sqrt{\text{B.M.}^2 + \text{T.M.}^2} \quad \ldots \quad (\text{VII})$$

It will be found that equation (VII), giving the so-called equivalent twisting moment, is the one most generally applied. It should be noted, however, that the stress concerned here is a *tensile* one and not a shear as in a proper twisting moment.

Example IV.—Investigate an expression in terms of f_t, f', and f_s, which will give the resultant tensile stress, f', per square inch of section in a material which is subjected at the same time to a direct tensile stress of f_t lbs. per square inch, and to a shearing stress, f_s, lbs. per square inch. A bar of iron is at the same time under a direct tensile stress of 5,000 lbs. per square inch, and to a shearing stress of 3,500 lbs. per square inch. What would be the resultant equivalent tensile stress in the material? (S. & A. Hons. Exams., 1896.)

Answer.—The complete investigation referred to in the first part of this question is given in the text, and equation (V) is the expression required. It only remains to find the numerical value of f', having given

$$f_t = 5{,}000 \text{ and } f_s = 3{,}500$$

$$\therefore \quad f' = \frac{5{,}000}{2} + \sqrt{\frac{5{,}000^2}{4} + 3{,}500^2}$$

$$,, \quad = \mathbf{6{,}800 \text{ lbs. per sq. in. fully.}}$$

Example V.—A wrought-iron shaft is subjected simultaneously to a bending moment of 8,000 inch-lbs., and to a twisting moment of 15,000 inch-lbs. Find the twisting moment equivalent to these two, and the least safe diameter of the shaft. The safe stress against shearing is to be taken at 8,000 lbs. per square inch. Prove clearly the formula you employ. (S. & A. Hons. Exam., 1890.)

Answer.—Here we have:—

$$\text{B.M.} = 8{,}000 \text{ inch-lbs.}$$

And $\quad \text{T.M.} = 15{,}000 \quad ,,$

Hence, by formula (VII) we get:—

$$\text{T.M.}' = 8{,}000 + \sqrt{8{,}000^2 + 15{,}000^2}$$

$$,, \quad = \mathbf{25{,}000 \text{ inch-lbs.}}$$

To find the diameter of the shaft to withstand this T.M.′ with a shearing stress of not over 8,000 lbs. per square inch, we employ formula (II) making:—

$$\text{T.M.} = \text{T.R.} = \frac{\pi}{16} D^3 f.$$

$$\therefore \quad D = \sqrt[3]{\frac{\text{T.M.}'}{\frac{\pi}{16} \cdot f}} = \sqrt[3]{\frac{25{,}000}{\frac{3\cdot1416}{16} \times 8{,}000}} = \mathbf{2\cdot51 \text{ inches.}}$$

Stiffness of Shafts.—Angle of Twist.—We have already seen that the effect of a turning moment applied to a shaft is to twist one part relatively to another. Hitherto we have been dealing only with the resistance the shaft offers to being twisted—that is to say, we have been concerned only with the *strength* of the shaft without regard to the question of *stiffness*. In many cases

—especially in light machinery—the question of the stiffness of the shafting is of greater importance than that of strength.

The stiffness of a shaft is measured by the smallness of the angle of twist per unit length of the shaft.

Turning back to the figure illustrating strain in a shaft, let dl be the axial distance, in inches, between the two sections whose diameters are A B, $a\,b$, and let $d\theta$ be the circular measure of the angle between those diameters when the shaft is twisted; then the torsional, or shearing *strain* at the surface of the shaft, is

$$= \left(\frac{D}{2}\right) \times \frac{d\theta}{dl}.$$

D, as before, being the extreme diameter of the shaft in inches,

Let f = Surface stress in the material of the shaft in lbs. per sq. inch.

,, C = Modulus or coefficient of shearing elasticity or of rigidity in lbs. per sq. inch.

Then, since

$$C = \frac{\text{stress}}{\text{strain}} = \frac{f}{\left(\frac{D}{2}\right) \cdot \frac{d\theta}{dl}}.$$

$$\therefore \qquad d\theta = \frac{2f}{C\,D} \cdot dl.$$

Hence, for a shaft L inches long we have, by a simple integration, the angle of twist.

$$\theta = \frac{2f}{C\,D}\int_0^L dl = \frac{2fL}{C\,D}.$$

To express this result in terms of the twisting moment and the diameter of the shaft, we have :—

$$f = \frac{\text{T.M.}}{\frac{\pi}{16} D^3} \text{ for solid shafts.}$$

And,

$$f = \frac{\text{T.M.}}{\frac{\pi}{16} \frac{D^4 - d^4}{D}} \text{ for hollow shafts.}$$

Making these substitutions and simplifying, we get :—

Angle of twist for solid shafts,

$$\left.\begin{aligned} \theta &= \frac{10{\cdot}2\,(\text{T.M.})\,\text{L}}{\text{C}\,\text{D}^4}\text{. radians.} \\ \text{Or,}\qquad \theta^\circ &= \frac{584\,(\text{T.M.})\,\text{L}}{\text{C}\,\text{D}^4}\text{. degrees.} \end{aligned}\right\} \quad . \ . \ . \ \text{(VIII)}$$

And, for hollow shafts,

$$\left.\begin{aligned} \theta &= \frac{10{\cdot}2\,(\text{T.M.})\,\text{L}}{\text{C}\,(\text{D}^4 - d^4)}\text{. radians.} \\ \text{Or,}\qquad \theta^\circ &= \frac{584\,(\text{T.M.})\,\text{L}}{\text{C}\,(\text{D}^4 - d^4)}\text{. degrees.} \end{aligned}\right\} \quad . \ . \ . \ . \ \text{(IX)}$$

By the equations just established, we see that, while the strength of shafts vary as the *third* power of their diameters, their stiffness varies as the *fourth* power.

Example VI.—Establish a formula for the moment of resistance to torsion of a solid shaft of circular section. The angle of torsion of a shaft is limited to 1° for each 10 feet of length; find the diameter of a solid round shaft to transmit 100 H.P. at 50 revolutions per minute, the modulus of resistance to torsion being 10,000,000 lbs. per sq. inch. (S. & A. Hons. Exam., 1892.)

Answer :—

Here, $\theta = 1^\circ$ when, $\text{L} = 10 \times 12 = 120$ inches

And, $\text{C} = 10{,}000{,}000.$

Also, $$\text{T.M.} = 63{,}024 \times \frac{\text{H.P.}}{n} = 63{,}024 \times \frac{100}{50}$$

$$\text{,,} \quad = 126{,}048 \text{ inch-lbs.}$$

Now, applying formula (VIII) the given conditions are that:—

$$1^\circ = \frac{584 \times 126{,}048 \times 120}{10{,}000{,}000 \times \text{D}^4}$$

Hence, solving for D, we get :—

$$\text{D} = \sqrt[4]{\frac{584 \times 126{,}048 \times 120}{10{,}000{,}000}} = 5{\cdot}45 \text{ inches.}$$

LECTURE XXX.—QUESTIONS.

1. A 10-inch shaft has a 4-inch hole run through it; what fraction of its weight is removed? To what extent is its strength in resisting torsion affected? *Ans.* 16 per cent.; 2·5 per cent. nearly.

2. A hollow shaft is 10 inches external diameter and 4 inches internal diameter; compare its strength to resist torsion with that of a solid shaft of the same weight.

3. Cylindrical bars of metal, each of 1 inch diameter, are exposed to torsion by weights applied at the end of a 12-inch lever. What would be the probable ultimate strength in the case of good specimens of wrought iron and cast iron. State the law according to which the strength of shafting increases by increasing its diameter.

4. If a wrought-iron shaft of 1 inch diameter is broken by the torsion of a load of 800 lbs. acting at the end of a 12-inch lever, find the weight which, when applied to the end of the same lever, would break a shaft of the same material, but 3 inches in diameter. State, in general terms, the reasoning by which you arrive at the result. (S. and A. Exam., 1891.) *Ans.* 21,600 lbs.

5. If a shaft of 3 inches diameter transmits safely 33 horse-power at 100 revolutions per minute, what size of shaft will transmit safely 20 horse-power at 150 revolutions per minute. *Ans.* 2·22 inches.

6. If 800 lbs. at the end of a 12-inch lever be a safe stress to apply to a wrought-iron bar 1 square inch in section, find the effort which a shaft 2 inches in diameter can transmit at the circumference of a pulley one foot in diameter, and making 300 revolutions per minute. Find also the horse-power transmitted. *Ans.* 8,893 lbs.; 254 H.P.

7. A shaft is of given material and given diameter, find an expression for the moment of resistance to torsion. Given the maximum stress to which the material may be subjected, find the diameter of a shaft which will transmit a given horse-power at a given number of revolutions per minute.

8. A twisting moment of 9,600 inch-pounds is sufficient to break a wrought-iron shaft of 1 inch diameter. Use 6 as a factor of safety, and hence determine what horse-power can be safely transmitted through a shaft of 3 inches diameter when running at 120 revolutions per minute. Prove the formula which you employ. (S. & A. Hons. Exam., 1889.)

9. Investigate an expression for the moment of resistance to torsion of a given cylindrical shaft when subjected to a given twisting moment. What is the maximum horse-power which could be transmitted by a shaft 3 inches in diameter when making 150 revolutions per minute, it being given that the shearing stress in the material is not to exceed 7,500 lbs. per square inch? (S. & A. Hons. Exam., 1887.) *Ans.* 94·5 H.P.

10. If θ be the angle of twist expressed in circular measure in a length of shafting l, M the twisting moment, C the modulus of transverse elasticity, and d the diameter of the shaft, prove that—

$$\theta = \frac{10{\cdot}2\,\mathrm{M}\,l}{\mathrm{C}\,d^4}.$$ (S. & A. Hons. Exam., 1893.)

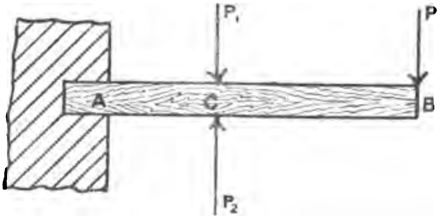
LECTURE XXXI.

CONTENTS.—Strength of Beams and Girders—Definitions of Shearing Force and Bending Moment—Beam Fixed at one end and Loaded at the other—Beam Fixed at one end and Loaded Uniformly—Beam Supported at both ends and Loaded in the middle—Example I.—Beam Supported at ends and Loaded anywhere — Beam Supported at both ends and Loaded Uniformly—Examples II. and III.—Floating Beams—Travelling Loads—Two Loads Moving at a Fixed Distance apart—Example IV.—Distributed Travelling Load—Questions.

Strength of Beams and Girders.—The subject under this heading is one that naturally divides itself into two portions. (1) The determination of the resultant effects of the applied loads at any section of a beam or girder; and (2) the nature and amount of the resistance offered by the beam or girder to rupture at that section.

When the section under consideration is in the same plane as the load, the only effect the load has at that section is a tendency to *shear* the beam; but in the more general case, where the load acts at a distance from the given section, we have, in addition, a tendency to curve or bend the beam at the section. Hence the name *Bending Moment* is given to this latter effect.

In the accompanying figure, let A B represent a cantilever or beam fixed at one end, with a load P applied at the free end; and let C be any section in the beam. At C let there be applied two equal and opposite forces P_1, P_2, of the same magnitude as P. The introduction of these forces does not affect the equilibrium of the system, as P_1 and P_2 balance each other. Hence, the effect of P at the section C is equivalent to that of a



ILLUSTRATING SHEARING AND BENDING ACTION.

couple $P P_2$, with a single force P_1. A general proof of this important theorem is given in Vol. I., Lecture III., Prop. II. The couple constitutes the **Bending Moment** (B.M.), and the single force P_1, the **Shearing Force** (S.F.) at the section C.

Definition.—**The Shearing Force at any section of a beam is the algebraic sum of all the forces acting on either side of that section.**

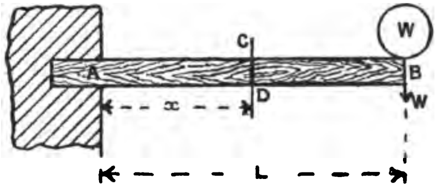
Definition.—**The Bending Moment at any section is the algebraic sum of the moments of all the forces acting on either side of that section.**

Or, in symbols, if P denote any one of the forces acting on one side of a section, and at a distance x, from it; consider all the forces on the same side of the section as P, paying due regard to their sign—that is, if we reckon forces acting upwards as positive, we must regard those acting downwards as negative.

$$\left.\begin{array}{lr}\text{Then,} & \text{S.F.} = \Sigma P. \\ \text{And,} & \text{B.M.} = \Sigma P x.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I})$$

Beam Fixed at one end and Loaded at the other.—Let C D be a cross-section anywhere within the length of the beam at a distance of x inches from the fixed end A. To find the S.F. and B.M. at C D, we observe that the only force acting to the right of the section is W lbs. Hence:—

$$\text{S.F.} = W \text{ lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II})$$



Beam Fixed at one end, Loaded at other.

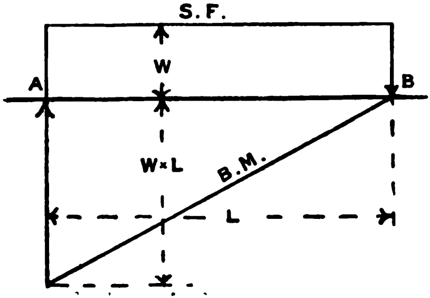
It is independent of x, and therefore the same for all such sections as C D.

The B.M. at C D is W multiplied by its distance from the section in inches. Hence:—

$$\text{B.M.} = W \times BD = W(L - x) \text{ inch-lbs.} \quad . \quad . \quad . \quad (\text{III})$$

This equation is true whatever may be the position of W on the beam, so long as L denotes its distance in inches from the fixed end, and C D is between W and the support.

In this case, the diagram of the S.F. is a straight line parallel to the base and at a distance of W lbs. from it. Since (III) is the equation of a straight line, the B.M. is therefore a quantity increasing uniformly from zero, where $x = L$, to W L inch-lbs., where $x = 0$, as shown by the accompanying figure.



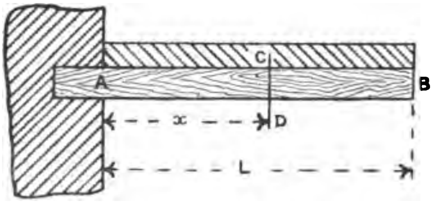
S.F. curve is a straight line.

B.M. curve is a straight line.

DIAGRAM OF S.F. AND B.M. FOR BEAM FIXED AT ONE END AND LOADED AT THE OTHER.

Beam Fixed at one end, and Loaded Uniformly.—Let the load on the beam be w lbs. per inch-run, it is required to find the shearing force and bending moment at any section C D, at x inches from the fixed end. As before, consider the part of the beam to the right of C D. The only force is the weight of that portion of the load carried by B D, so that :—

$$\text{S.F.} = w \times \text{B D} = w\,(\text{L} - x) \text{ lbs.} \quad . \quad . \quad . \quad (\text{IV})$$



BEAM FIXED AT ONE END AND LOADED UNIFORMLY.

The moment of the portion of the load on B D with respect to C D is the same as if it were all concentrated at the middle point of B D. Hence :—

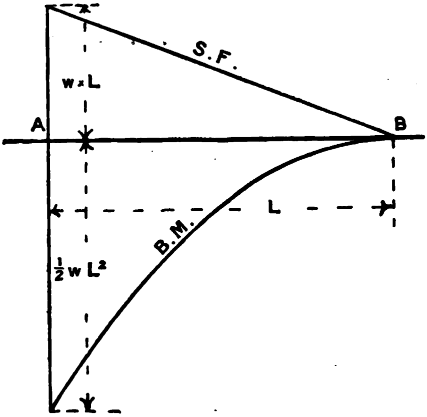
$$\text{B.M.} = w \times \text{B D} \times \tfrac{1}{2}\,\text{B D} = \tfrac{1}{2}\,w \times \text{B D}^2 = \tfrac{1}{2}\,w\,(\text{L} - x)^2 \text{ inch-lbs.} \quad (\text{V})$$

Equations (IV) and (V) show us that both the **S.F.** and **B.M.** vanish when $x = \text{L}$; and when $x = 0$, we get :—

$$\text{S.F.} = w\,\text{L lbs.} \qquad (\text{IV}_a)$$

And,
$$\text{B.M.} = \tfrac{1}{2}\,w\,\text{L}^2\ \text{inch-lbs.} \qquad (\text{V}_a)$$

The diagrams of S.F. and B.M. for this case take the forms shown in the accompanying figure.



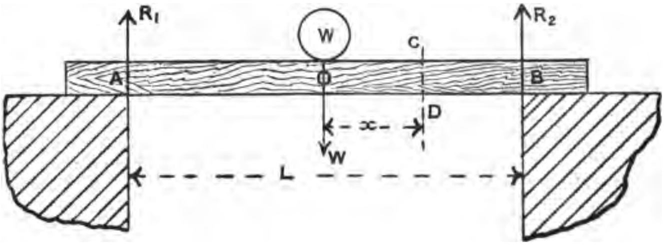
S.F. curve is a straight line.

B.M. curve is a parabola with vertex at B.

Diagram of S.F. and B.M. for Beam Fixed at one end and Loaded Uniformly.

Beam Supported at both ends, and Loaded at the Middle.—In this case we measure x from the middle point of the beam. Since W is equidistant from A and B, the reactions at those points, R_1 and R_2, are equal to each other, and since their sum is W, we have:—

$$R_1 = R_2 = \tfrac{1}{2}\,W\ \text{lbs.}$$



Beam Supported at both ends and Loaded at Middle.

The only force to the right of C D is R_2, and its leverage is B D.

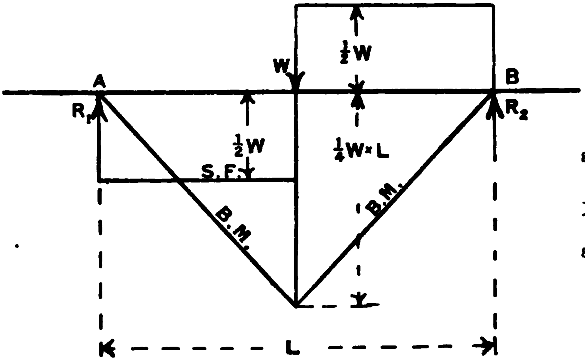
Hence, $\text{S.F.} = R_2 = \frac{1}{2} W$ lbs. . . . . (VI)

And, $\text{B.M.} = R_2 \times BD = \frac{1}{2} W (\frac{1}{2} L - x)$ inch-lbs. . (VII)

Here, the B.M. vanishes when $x = \frac{1}{2}$ L, and increases uniformly from this until $x = 0$, when it attains its maximum value, $\frac{1}{4}$ W L inch-lbs.

Or, **Maximum B.M. = $\frac{1}{4}$ W L inch-lbs.** . . . (VII_a)

The following figure shows the diagrams of S.F. and B.M. for this case:—



S.F. curve is a pair of straight lines.

B.M. curve is a pair of straight lines.

DIAGRAM OF S.F. AND B.M. FOR BEAM SUPPORTED AT BOTH ENDS AND LOADED IN MIDDLE.

EXAMPLE I.—In a beam of length L, supported at both ends and loaded at the middle with a load W, show that the bending moment is greatest at the centre of the beam and equal to $\frac{1}{4}$ W L. Then determine graphically the bending moment and shearing force at a point 6 ft. from one support in a beam of 25 ft. span loaded with 5 tons at its centre. (Adv. S. & A. Exam., 1890.)

ANSWER.—We have already seen from equation (VII) that for a beam loaded as in this example, the B.M. at any distance x, from its middle point, is:—

$$\text{B.M.} = \tfrac{1}{2} W (\tfrac{1}{2} L - x).$$

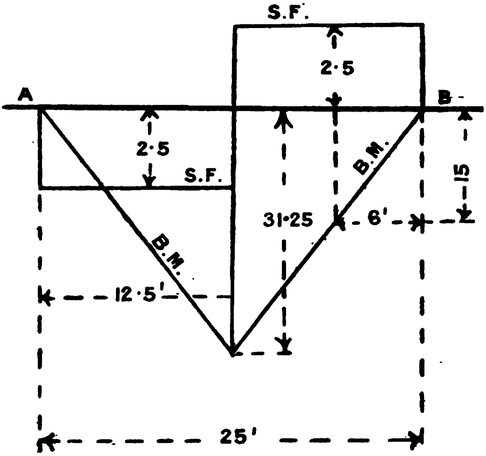
This is obviously greatest when $x = 0$—that is, at the centre. Then:—

$$\text{Maximum B.M.} = \tfrac{1}{4} W L; \text{ and S.F.} = \tfrac{1}{2} W.$$

For the values of W and L given in the example, we get:—

$$\textbf{Maximum B.M.} = \tfrac{1}{4} \times 5 \times 25 = 31{\cdot}5 \text{ ft.-tons.}$$

And, $$\textbf{S.F.} = \tfrac{1}{2} \times 5 = 2{\cdot}5 \text{ tons.}$$



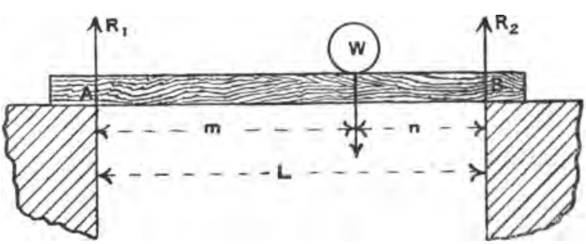
S.F. AND B.M. CURVES FOR EXAMPLE I.

The accompanying figure shows the diagrams of B.M. and S.F. as constructed from these data.

At 6 feet from one end the B.M. measures 15 ft.-tons. This is easily verified by means of the formula for B.M., because $x = 12{\cdot}5 - 6 = 6{\cdot}5$.

$$\therefore \qquad \text{B.M.} = \tfrac{1}{2} \times 5 \times (12{\cdot}5 - 6{\cdot}5) = 15 \text{ ft.-tons.}$$

Beam Supported at both ends, and Loaded Anywhere.—With a single concentrated load, the maximum bending moment will



BEAM SUPPORTED AT BOTH ENDS, AND LOADED ANYWHERE.

always occur immediately under the load, whether it be at the middle of the beam or not.

For the B.M. at any section at a distance x, from one end is $R \times x$, and this is greatest when x is largest; that is, when the section is under the load.

To find the reactions at the supports, we take moments about A and B, and get $R_2 \times L = W \times m$.

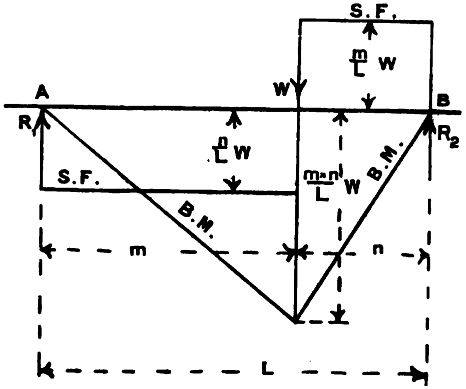
$\therefore R_2 = \frac{m}{L} W$ lbs. and $R_1 = \frac{n}{L} W$ lbs. These are the values of the S.F. to the right and left of W respectively.

$$\left.\begin{aligned} \text{S.F. (to right)} &= \frac{m}{L} W \text{ lbs.} \\ \text{S.F. (to left)} &= \frac{n}{L} W \text{ lbs.} \end{aligned}\right\} \quad . \; . \; . \quad \text{(VIII)}$$

Multiplying the first of these equations by n, or the latter by m, we get:—

$$\text{Maximum B.M.} = \left(\frac{m\,n}{L} W\right) \text{inch-lbs.} \quad . \; . \; . \quad \text{(IX)}$$

We can now construct the diagrams of S.F. and B.M.:—



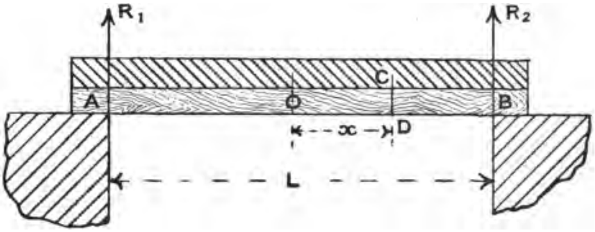
S.F. curve is two straight lines.

B.M. curve is two straight lines.

Diagram of S.F. and B.M. for Single Load in any position.

Beam Supported at both ends and Loaded Uniformly.—As before, let the weight per inch-run be denoted by w, then the total load carried by the beam will be w L lbs., and the reactions

R_1 and R_2, will each be $\frac{1}{2}wL$ lbs. Taking the forces to the right of the section C D.



Beam Supported at both ends and Loaded Uniformly.

$$\text{We get, S.F.} = R_2 - w \times BD = \tfrac{1}{2}wL - w(\tfrac{1}{2}L - x) = wx \text{ lbs.} \quad \text{(X)}$$

$$\begin{aligned} \text{And,} \quad \text{B.M.} &= R_2 \times BD - w \,.\, BD \times \tfrac{1}{2}BD \\ \text{,,} \quad &= \tfrac{1}{2}wL \times BD - \tfrac{1}{2}w \,.\, BD^2 \\ \text{,,} \quad &= \tfrac{1}{2}w \,.\, BD(L - BD) \\ \text{,,} \quad &= \tfrac{1}{2}w(\tfrac{1}{2}L - x)(\tfrac{1}{2}L + x). \end{aligned}$$

$$\therefore \quad \text{B.M.} = \tfrac{1}{2}w(\tfrac{1}{4}L^2 - x^2) \text{ inch-lbs.} \quad . \; . \; . \; . \; . \; . \quad \text{(XI)}$$

The limiting values of S.F. and B.M. are :—

When, $x = \frac{1}{2}L$; then, S.F. $= \frac{1}{2}wL$ lbs.; and, B.M. $= 0$. (X_a)

When, $x = 0$; then, S. F. $= 0$; and :—

$$\text{Maximum, B. M.} = \tfrac{1}{8}wL^2 \text{ inch-lbs.} \quad . \; . \quad (XI_a)$$

Plotting our diagrams of S.F. and B.M., we get the figure shown on next page.

When a beam carries more than one load, or is loaded in more ways than one, the simplest and safest way is to consider each load separately, without regard to the others, and then combine the separate effects so as to obtain the resultant action, as in Example II.

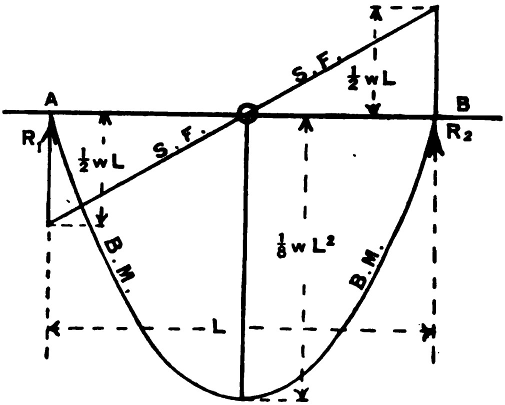
Example II.—Draw the bending moment and shearing force diagrams for a beam 12 feet long, supported at both ends, and loaded with weights of 4 and 6 tons at distances of 3 and 8 feet respectively, from one end of the beam. Explain fully the mode of arriving at these diagrams. (Adv. S. & A. Exam., 1887.)

Answer.—Measuring distances from the left end of the beam, and considering each load separately, we have, for the 4 tons, to the left of the load :—

$$\text{S.F.}_1 = \frac{n}{L} W = \frac{9}{12} \times 4 = 3 \text{ tons.}$$

And, to the right of it:—

$$\text{S.F.}_1 = \frac{m}{L} W = \frac{3}{12} \times 4 = 1 \text{ ton.}$$



S.F. curve is a straight line.

B.M. curve is a parabola with vertex below the middle of the beam.

DIAGRAM OF S.F. AND B.M. FOR A BEAM SUPPORTED AT BOTH ENDS AND LOADED UNIFORMLY.

The maximum B.M._1 due to this load is:—

$$\text{B.M.}_1 = \frac{m \times n}{L} W = \frac{3 \times 9}{12} \times 4 = 9 \text{ ft.-tons.}$$

It occurs immediately under the load.

Next taking the 6-tons load, we have to the left of it:—

$$\text{S.F.}_2 = \frac{n}{L} W = \frac{4}{12} \times 6 = 2 \text{ tons;}$$

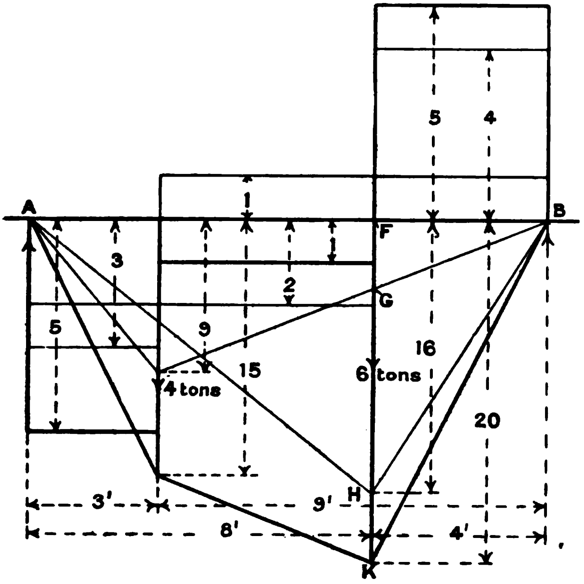
And to the right of it:—

$$\text{S.F.}_2 = \frac{m}{L} W = \frac{8}{12} \times 6 = 4 \text{ tons.}$$

The maximum B.M._2 due to the 6 tons is:—

$$\text{B.M.}_2 = \frac{m \times n}{L} W = \frac{8 \times 4}{12} \times 6 = 16 \text{ ft.-tons.}$$

Plotting these results, we get the accompanying figure:—



S.F. AND B.M. CURVES FOR EXAMPLE II.

The thin lines show the actions of the separate loads, and the full lines their combined results, obtained by taking the algebraic sum of the former.

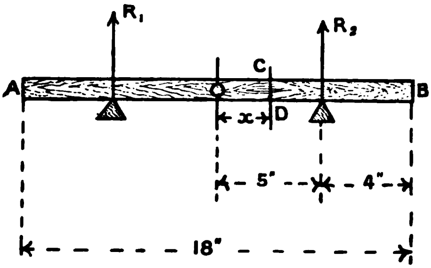
The student should here carefully observe the necessity of attending to the *sign* of the shearing force. Thus, between the weights we have a shearing force of 2 tons, which, on account of its sign, is drawn below the base line; also a shearing force of 1 ton drawn above the base line. The resultant shearing force between the loads is therefore the difference of these, and is drawn on the same side of the base line as the greater of its components.

The bending moments everywhere along the beam are of the same sign; therefore, to obtain the combined bending moment diagram, we have simply to add the ordinates of each separate diagram. Thus, to get the total bending moment at the section under the 6 tons load, we add F G (viz., that due to the 4 tons at that point) to F H (that due to the 6 tons). The result F K is therefore the total B.M. at that point.

It is quite sufficient to do this for the sections under each load, and then to join each of the points so obtained with each other and with the ends of the beam by straight lines. If drawn to scale, the **B.M.** at any other point can then be obtained by measuring the corresponding ordinate.

EXAMPLE III.—A horizontal uniform bar, 18 inches long, is laid over two supports, each 4 inches from its ends. Find two points at which the bending moments are zero. (Hons. S. & A. Exam.)

ANSWER.—Let w be the weight in lbs. per inch-run of the bar. Then the total weight of the bar will be $18w$ lbs., and the reactions will each be $9w$ lbs.



ILLUSTRATING EXAMPLE III.

Taking moments to the right of the section C D at a distance x inches from the centre of the bar, we get:—

$$\begin{aligned} \text{B.M.} &= R_2(5 - x) - w \,.\, BD \times \tfrac{1}{2} BD \\ ,, &= 9w(5 - x) - \tfrac{1}{2} w \,.\, BD^2 \\ ,, &= 9w(5 - x) - \tfrac{1}{2} w(9 - x)^2 \\ ,, &= \tfrac{1}{2} w(9 - x^2) \text{ inch-lbs.} \end{aligned}$$

The **B.M.** will be zero when $9 - x^2 = 0$; *i.e.*, when $x = \pm 3$ inches.

Hence, the required points are 3 inches on each side of the centre, or 2 inches inside of the supports.

Floating Beams.—When a *solid* body, such as a piece of wood of *uniform density*, floats in *still* water its weight and its buoyancy, or the resultant upward pressure of the water on the body, will at all points balance each other. There are consequently no shearing or bending stresses on the body, and each part is in equilibrium independently of the other parts.

But whenever those conditions are departed from, such as (1) when the floating body carries weights; (2) when it is not of uniform density, due to want of homogeneity in its material, if solid, or to its being hollow, or of a boat form; or (3) when it crosses waves, then bending and shearing stresses are set up.

Consider the case of a uniform beam of wood of rectangular section floating in still water. The beam will displace an amount of water exactly equal to its own weight. This is true, not only for the beam as a whole, but also for every individual segment of the beam. Any segment of the beam will displace just as much, and no more, water than it would do if floating by itself. The beam, therefore, is as free from stress as it would be if it were lying on a perfectly flat surface.

Suppose now that a weight W, be placed on the middle of a floating beam. This will cause the beam to sink to a greater depth and displace an *extra* volume of water. The *weight* of this *extra* displacement is exactly equal to W. What, now, is the condition of the beam as regards straining forces? Evidently, we need only consider the weight W, and the extra displacement, due to its being carried by the beam; because the upward reaction of the displacement due to the beam's own weight, is still at all points balanced by the downward weight of the beam. In other words, the condition of the beam, so far as its own weight and displacement are concerned, is in no way affected by the addition of the load.

To give definiteness to our ideas, let W be expressed in lbs., and let L denote the length of the beam in inches.

Then the forces we have to consider are:—

(1) W lbs. concentrated at the middle of the beam and acting downwards.

(2) The displacement of W lbs. of water uniformly distributed along the whole length of the beam and acting upwards, with an intensity of $\frac{W}{L}$ lbs. per inch of length.

The case is, therefore, analogous to that of a beam uniformly loaded and supported at its centre; or what is virtually the same thing, two beams of length equal to $\frac{1}{2}$ L, fixed at one end and loaded uniformly. For, in order to obtain the shearing force and the bending moment at any section of the beam, x inches to either side of W, we have simply to substitute $\frac{W}{L}$ for w, and $\frac{1}{2}$ L for L in equations (IV) and (V), and we get:—

$$\text{S.F.} = \frac{W}{L}\left(\tfrac{1}{2}L - x\right) \text{ lbs.} \quad . \quad . \quad . \quad . \quad (\text{XII})$$

And, $$\text{B.M.} = \frac{W}{2\,L}(\tfrac{1}{2}\,L - x)^2 \text{ inch-lbs.} \quad . \quad . \quad . \quad (\text{XIII})$$

Under W the shearing force and bending moment are each a maximum. Their values may be found by making $x = 0$.

Then the $$\textbf{Maximum S.F.} = \tfrac{1}{2}\,W \text{ lbs.} \quad . \quad . \quad . \quad (\text{XII}_a)$$

And the $$\textbf{Maximum B.M.} = \tfrac{1}{8}\,W\,L \text{ inch-lbs.} \quad . \quad . \quad (\text{XIII}_a)$$

The diagrams of S.F. and B.M. for this case are constructed in identically the same way as for a beam fixed at one end and carrying a uniform load, but taking $\frac{1}{2}$ L as a base line instead of L.

Suppose that, instead of one weight in the middle, the beam is loaded with two weights, one at each end, and each equal to W lbs., it is easy to see that the condition now is that of a beam uniformly loaded with $\frac{2\,W}{L}$ lbs. per inch-run and supported at each end. We have, therefore, only to apply formulæ (X) and (XI), substituting $\frac{2\,W}{L}$ for w, when we get:—

$$\text{S.F.} = \frac{2\,W}{L} \cdot x \text{ lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV})$$

And, $$\text{B.M.} = \frac{W}{L}(\tfrac{1}{4}\,L^2 - x^2) \text{ inch-lbs.} \quad . \quad . \quad . \quad . \quad (\text{XV})$$

Here the shearing force is a maximum when $x = \frac{1}{2}$ L, and the bending moment a maximum when $x = 0$.

Or, $$\textbf{Maximum S.F.} = W \text{ lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV}_a)$$

And, $$\textbf{Maximum B.M.} = \tfrac{1}{4}\,W\,L \text{ inch-lbs.} \quad . \quad . \quad (\text{XV}_a)$$

The diagrams of S.F. and B.M. are, therefore, in every way similar to those for a uniformly loaded beam supported at the ends.

Travelling Loads.—The simplest case of a movable load is that, wherein we are given a weight, say a heavy cylindrical body, rolling along a beam, to find the equations of maximum S.F. and B.M. for any position of the load, and exhibit these results in a diagram.

Referring to formulæ (VIII) and (IX), and the diagrams already deduced for a fixed load in *any* position on a beam, we have for the maximum S.F. to the immediate right of the load :—

$$\text{S.F.} = \frac{m}{\text{L}}\,\text{W lbs.}$$

And, to the immediate left of the load :—

$$\text{S.F.} = -\frac{n}{\text{L}}\,\text{W lbs.}$$

For the maximum B.M., which occurs immediately under the load :—

$$\text{B.M.} = \frac{m\,n}{\text{L}}\,\text{W inch-lbs.}$$

Putting $m = x$ so that $n = \text{L} - x$, we obtain, when the load is x inches from the left end of the beam :—

$$\left.\begin{array}{ll}\textbf{The Maximum S.F.}\ \textit{(just to right of the section)} & = \frac{\text{W}}{\text{L}}\,x \\ \quad ,, \qquad ,, \qquad \textit{(just to left of the section)} & = \frac{\text{W}}{\text{L}}(x-\text{L})\end{array}\right\}\text{(XVI)}$$

$$\textbf{And,}\qquad \textbf{Maximum B.M.} = \frac{\text{W}}{\text{L}}(\text{L} - x)\,x. \;.\;.\;.\;. \quad \text{(XVII)}$$

To construct the diagram of S.F., we observe that its equation is that of a straight line, and that to the right of the section considered its value is zero when the load just starts from the left end of the beam, and increases uniformly as the load approaches the other end. That is :—

When, $x = 0$; then, S.F. $= 0$.

Also when, $x = \text{L}$; then, S.F. $=$ W lbs.

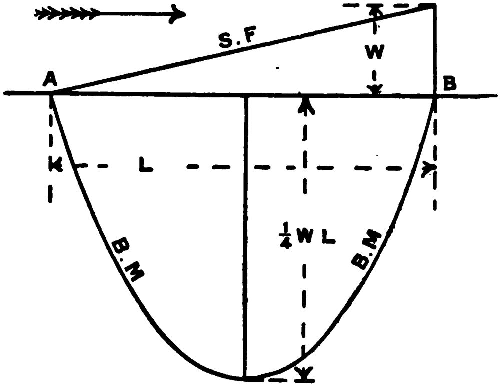
There is also another line for the shear to the left of the section. This line (which has been omitted in the diagram) passes through B, and its ordinate is $-\text{W}$ at the end A.

The equation of the B.M. curve is that of a parabola, whose axis is vertical, and passes through the middle point of the beam, where, of course, the maximum value of B.M. occurs. To construct this diagram, we have :—

When, $x = 0$, or $x = \text{L}$; then B.M. $= 0$.

Also when, $x = \frac{1}{2}\text{L}$; then B.M. $= \frac{1}{4}\,\text{W L}$ inch-lbs.

The arrow indicates the direction in which the load is supposed to move.



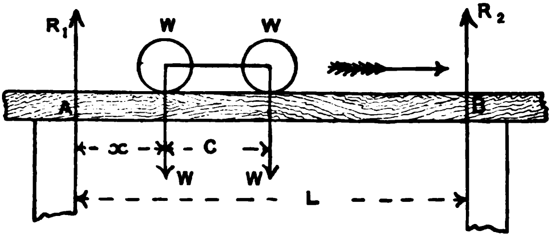
S.F. curve is a pair of straight lines (one of which has been omitted in the diagram).

B.M. curve is a parabola with axis vertical.

Diagrams of S.F. and B.M. for Rolling Load.

Two Loads moving at a fixed distance apart.—From the above simple case we may easily pass to a very important practical example of moving loads—viz., overhead travelling cranes.

Here the crane rests on a carriage with four wheels running on two rails carried by girders, the weight of the whole machine together with the load being equally distributed over the wheels. Hence, considering one girder only, our problem is reduced to that of two equal loads moving along the girder at a fixed distance apart.



Illustrating Travelling Crane Problem.

In the figure let W be the weight resting on each wheel, and c be the distance between their centres. If the motion be supposed to be in the direction shown by the arrow, it is evident that until the carriage gets to the middle of the girder the maximum shearing force and bending moment will occur under the leading wheel—that is, if we estimate the shearing force to the immediate right of the wheel. But as the same thing takes place in the reverse order when the carriage moves from the other end of the girder in the opposite direction, we shall take the section of the girder immediately under the following wheel and estimate the shearing force and bending moment for that position. This method of procedure will be found to lead to simpler equations than if we had taken the leading wheel as our point of reference.

Now, considering the forces acting to the left of the wheel, we easily see :—

That, $$\text{S.F.} = R_1.$$

And, $$\text{B.M.} = R_1 \times x.$$

To find R_1 we take moments about B, which gives us :—

$$R_1 \times L = W\{L-(x+c)\} + W(L-x)$$

$$\text{,, \quad ,,} = W\{2(L-x)-c\}.$$

$$\therefore \qquad R_1 = \frac{W}{L}\{2(L-x)-c\}.$$

Hence, $$\text{S.F.} = \frac{W}{L}\{2(L-x)-c\}. \quad . \quad . \quad . \quad \text{(XVIII)}$$

And, $$\text{B.M.} = \frac{W}{L}\{2(L-x)-c\}x. \quad . \quad . \quad . \quad \text{(XIX)}$$

The equation for the S.F. is that of a straight line, and for the B.M. a parabola. To find the position and dimensions of those diagrams, we see that :—

When $x = 0$, S.F. $= \frac{W}{L}(2L - c)$, and B.M. $= 0$.

Again, both S.F. and B.M. will vanish when $2(L-x) - c = 0$; that is, when $x = L - \frac{c}{2}$.

To find the maximum ordinate of the B.M. curve, we have the condition that, when the B.M. is a maximum :—

$$\frac{d}{dx}(\text{B.M.}) = 0.$$

That is, $$\frac{d}{dx}\{2(L-x)-c\}x = 0.$$

Or, $$\frac{d}{dx}\{(2\mathrm{L}-c)x - 2x^2\} = 0.$$

$$\cdot\ (2\mathrm{L} - c) - 4x = 0.$$

Hence, $$x = \frac{\mathrm{L}}{2} - \frac{c}{4}.$$

The shearing force diagram will, therefore, consist of two straight lines cutting each other at the middle of the girder's length, and the bending moment diagram will consist of two equal parabolas also intersecting at the middle of the girder. The axes of those equal parabolas will be equidistant from the middle of the girder and $\frac{1}{2}c$ units apart.

The following numerical example will elucidate this important case much better than a bare examination of formulæ:—

Example IV.—In a travelling crane of 40 feet span the load is supported on a carriage which runs upon two similar girders, the axles of the carriage being 8 feet apart, and a load of $2\frac{1}{2}$ tons coming upon each wheel. Obtain a diagram showing the maximum bending moment at every section of the girder, and give the numerical values at distances of 10, 15, and 20 feet from one end. (Hons. S. & A. Exam., 1880.)

Answer.—Applying our general formulæ, we have, for the bending moment at any distance x ft., from one end:—

$$\text{B.M.} = \frac{\mathrm{W}}{\mathrm{L}}\left[(2\mathrm{L}-c) - 2x\right]x.$$

Here, $\mathrm{W} = 2\cdot5$ tons, $\mathrm{L} = 40$ ft., and $c = 8$ ft.

$$\therefore \quad \text{B.M.} = \frac{2\cdot5}{40}\left[80 - 8 - 2x\right]x.$$

Or, $$\text{B.M.} = \tfrac{1}{8}(36 - x)x \text{ ft.-tons.}$$

For the numerical values asked for, we have:—

When $x = 10$ ft.; B.M. $= \frac{1}{8}(36 - 10)10 =$ **32·5 ft.-tons.**

When $x = 15$ ft.; B.M. $= \frac{1}{8}(36 - 15)15 =$ **39·375 ft.-tons.**

When $x = 20$ ft.; B.M. $= \frac{1}{8}(36 - 20)20 =$ **40 ft.-tons.**

We have seen, that the B.M. attains its maximum value when:—

$$x = \frac{\mathrm{L}}{2} - \frac{c}{4} = \frac{40}{2} - \frac{8}{4} = 18 \text{ ft.}$$

Hence, the maximum B.M. $= \frac{1}{8}(36 - 18)18 =$ **40·5 ft.-tons.**

The S.F. is not asked for in the question, but we here add it so as to make the example more complete.

The maximum S.F. occurs when $x = 0$, and when $x = \mathrm{L}$ its value for this case then being:—

$$\text{Maximum S.F.} = \frac{2\cdot5}{40}(80-8) = 4\cdot5 \text{ tons.}$$

And, like the B.M., it is zero when $x = 40 - \frac{8}{2} = 36$ feet.

The following figure shows the S.F. and B.M. diagrams as required for this example.

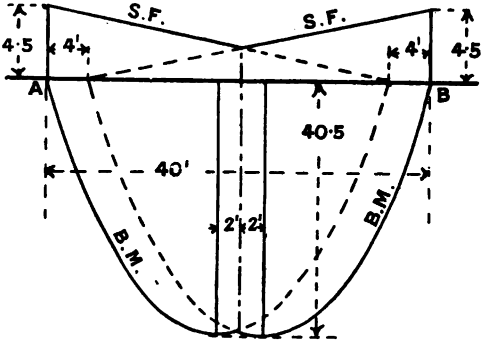
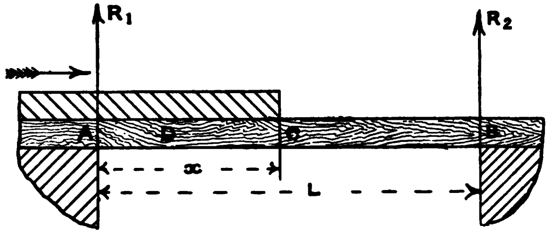


DIAGRAM OF S.F. AND B.M. FOR A TRAVELLING CRANE.

Distributed Travelling Load.—The last case we shall consider is that in which a continuous load of uniform density, and long enough to completely cover it, comes on to a girder and moves off at the other end, such as a long train of uniform weight passing over a bridge.



ILLUSTRATING TRAVELLING LOAD OF UNIFORM INTENSITY.

In the figure, let w denote the load per unit of length. When the load is in the position shown, it is clear that the S.F., at all points to the right of C, will be equal to R_2; and that at any section D, to the left of C, the S.F. will be less than R_2 by the weight of the portion of the load covering C D. It at once follows that the S.F. is greatest at C, the front of the load, and this is true for all positions of C.

Hence, $\text{S.F.} = R_2.$

Taking moments about A, we have :—

$$R_2 \times L = wx \times \tfrac{1}{2}x.$$

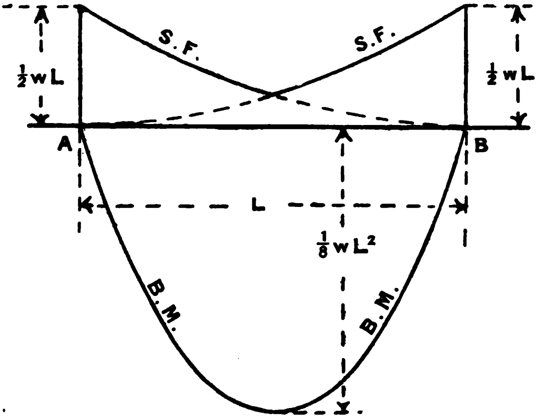
$$\therefore \qquad R_2 = \frac{wx^2}{2L}.$$

That is, $$\text{S.F.} = \frac{wx^2}{2L} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(XX)}$$

The shearing force is, therefore, proportional to the square of the length of the part of the load resting on the girder.

The curve of the maximum bending moment is very easily deduced in this case. We have only to remember that the B.M. at *any fixed* section in the girder will get greater and greater for every additional part of the load that comes upon it; so that when the girder is wholly covered by the load the B.M. at every position will then be a maximum. The B.M. diagram is therefore identical with that given for a beam loaded uniformly, whilst the S.F. diagram becomes a parabola instead of a straight line.

The following figure shows how the S.F. and B.M. diagrams are constructed for this case.



S.F. and B.M. Diagrams for Travelling Continuous Load of Uniform Intensity.

LECTURE XXXI.—QUESTIONS.

1. Define "*bending moment*" and "*shearing force.*" A uniform beam weighing 15 cwts. rests on supports at its ends 20 feet apart. The beam is loaded with three weights of 4, 6, and 10 cwts. at distances of 2, 7, and 12 feet respectively from one of the supports. Find the B.M. and S.F. at a point 8 feet from the same support. *Ans.* B.M. = 98 ft.-cwts.; S.F. = 3 cwts.

2. A bar of pine 44 inches long rests on props at its extremities, and just supports 10 weights, of 14 lbs. each, hung at equal intervals of 4 inches along the rod. Find the amount of a single weight, which, if hung at the centre of the bar, would strain it to the same extent. *Ans.* 43·27 lbs.

3. A batten of fir, 6 feet in length and supported at its extremities, will just sustain a load of 520 lbs. when hung at the centre. If this weight be removed, and two weights, each equal to P lbs., be hung at distances of 2 and 4 feet along the bar, what is the greatest value which may be assigned to P? *Ans.* 390 lbs.

4. A beam, 20 feet long, whose weight is neglected, is supported at both ends and loaded with 1 ton evenly distributed along its length. Find the bending moment at a distance of 7 feet from one end. *Ans.* 5,096 ft.-lbs.

5. A beam, whose weight may be neglected, rests on supports at its ends 15 feet apart. Weights of 10, 6, 5, and 12 cwts. rest on the beam at intervals of 3 feet apart, the weight of 10 cwts. being 3 feet from one support. Find the points where the maximum bending moment and shearing force occur, and obtain their values. Construct the diagrams of bending moments and shearing force for the whole beam. *Ans.* The *max.* B.M. = 66 ft.-cwts., and occurs at all points between the weights 6 and 5 cwts.; the *max.* S.F. = 17 cwts., and occurs at the point where the weight of 12 cwts. rests.

6. A uniform cantilever, or beam fixed at one end and free at the other, 10 feet long, weighs 6 cwts., and carries two loads, one of 2 cwts. at the free end, and the other 4 cwts. at its middle point. Construct the shearing force diagram for the whole cantilever, and find the shearing forces at points 2½ feet and 6 feet from fixed end. *Ans.* 10·5 cwts.; 6·4 cwts.

7. A block of wood weighing 800 lbs., 20 feet long and 12 inches square, floats in water, and is loaded—

(1) By a weight of 200 lbs., placed at each extremity;
(2) By a weight of 400 lbs. at the centre.

Show what forces act on the beam, and draw the curves of shearing force and bending moment for each case. (S. & A. Exam., 1892.)

8. A girder is supported at both ends, and has a clear span of 30 feet. Show by means of a curve the position and magnitude of the greatest bending moment produced by a load of 20 tons as it rolls from one end to the other of the girder. Obtain the numerical results for distances respectively of 10 and 15 feet from one end. (S. & A. Hons. Exam., 1895.)

9. Prove an algebraic formula to show that, with a continuous load of uniform intensity passing over a beam AB such as when a long train passes over a bridge A to B, the maximum shearing stress to any point K of the beam occurs when the part AK is fully loaded while the part

K B is entirely unloaded, and that the magnitude of the stress is proportional to the square of the distance of K from the point A. A train of 1 ton per foot run, and upwards of 100 feet in length, passes over a bridge of 100 feet span; what would be the maximum shearing stresses at distances of 25 and 50 feet respectively from one end of the bridge? Show how to determine graphically the shearing stresses in the beam. (S. & A. Hons. Exam., 1896.)

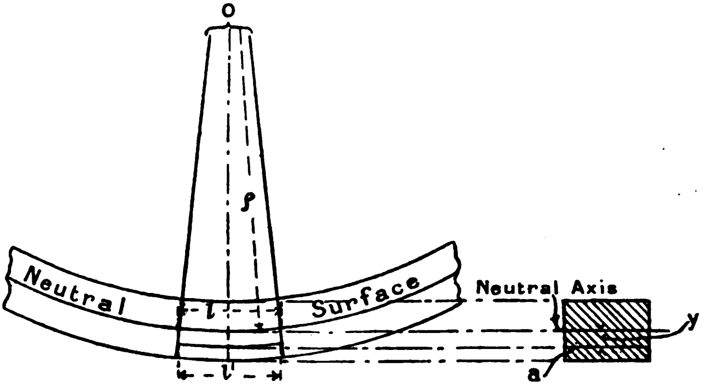
LECTURE XXXII.

CONTENTS.—Resistance of Beams to Flexure—Examples I., II., III., and IV.—Thin Wrought-Iron Girders—Example V.—Curvature and Deflection of Beams—Example VI.—Uniform Beam on Three Supports—Uniform Beam fixed at one end and supported at the other—Beams fixed at both ends and loaded at centre—Beams fixed at both ends and loaded uniformly—Tables—Questions.

Resistance of Beams to Flexure.—In the previous Lecture we saw that the effect of loading a beam was to give rise to both shearing and bending.

From the theory of couples set forth in Vol. I. we know that nothing but a couple can balance a couple. The resistance which a beam offers to bending must be of this nature, and therefore a couple of equal magnitude to that of the applied load, but of opposite tendency. The tendency of the applied couple is to bend or curve the beam, whilst the tendency of the induced couple is to oppose this curving action.

When a beam is curved the longitudinal fibres on the convex side of it are stretched beyond their normal length, and consequently they are in tension. On the concave side the fibres are shortened, and, therefore, they are in compression. Somewhere within the beam there must be a layer of fibres that are neither lengthened nor shortened, and are therefore unstressed.



ILLUSTRATING FLEXURE OF BEAMS.

This layer is termed the ***neutral surface*** of the beam, and the intersection of this surface with any cross-section of the beam is termed the ***neutral axis*** of that section. The neutral axis is of fundamental importance in the theory of beams, because it is the fulcrum about which both the bending and resisting couples act.

We shall now find the position of the neutral axis of any given section of a beam.

Let l be the length of a small portion of the neutral surface; l' that of a parallel layer of fibres on the stretched side of the beam, and at a distance y, from the neutral surface. If $l' = l$ when the beam is straight, it is evident that the amount of stretch in the fibres at distance, y, from the neutral surface will be $l' - l$, and the *strain* $\frac{l' - l}{l}$. Let ρ denote the radius of curvature of the neutral surface at the cross-section bisecting l. Then the radius of curvature corresponding to l' will be $= \rho + y$.

Hence,
$$\frac{\rho + y}{\rho} = \frac{l'}{l},$$

Or,
$$\frac{y}{\rho} = \frac{l' - l}{l}.$$

If f be the tensile stress at distance y, from the neutral axis, and E the modulus of elasticity of the material, we already know that :—

$$\frac{\text{stress}}{\text{strain}} = \text{E},$$

Or,
$$\frac{f}{\frac{l' - l}{l}} = \text{E}.$$

Substituting $\frac{y}{\rho}$ for $\frac{l' - l}{l}$, and inverting, we get :—

$$\frac{1}{\rho} = \frac{f}{\text{E}\,y}. \quad \ldots\ldots\ldots \text{(I)}$$

If we had considered in the same way a layer of fibres at a distance y', to the concave side of the neutral surface, and denoted the stress there as $-f'$ (the minus sign indicating compressive stress), we should have arrived at the equation :—

$$\frac{1}{\rho} = \frac{-f'}{\text{E}\,y'}. \quad \ldots\ldots\ldots (\text{I}_a)$$

Let a be a small element of the cross-sectional area at a distance y, then on the one side of the neutral axis we have for the total resistance to tension:—

$$\Sigma a f = \frac{E}{\rho} \Sigma a y. \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

On the other side of the neutral axis the total resistance to compression is:—

$$-\Sigma a f' = \frac{E}{\rho} \Sigma a y'. \quad . \quad . \quad . \quad . \quad . \quad (\text{II}_a)$$

But these forces constitute a couple, and are therefore equal. Hence, equating the right hand members, we have, neglecting the common factor, $\frac{E}{\rho}$:—$\Sigma a y = \Sigma a y'$.

The neutral axis, therefore, passes through the centre of area of each cross-section. If, however, E be not the same for Tensile and Compressive stresses, then the N.A. will not pass through the centre of the area, but will lie to the side having the greater value of E.

To obtain the magnitude of the resisting couple, we multiply the resistances, af and af', by their respective distances, y and y', from the neutral axis, and sum up these products for the whole section. Thus, from equation (II) the total moment of resistance on the convex side of the neutral axis is:—

$$\Sigma a f y = \frac{E}{\rho} \Sigma a y^2;$$

And on the concave side:—

$$-\Sigma a f' y' = \frac{E}{\rho} \Sigma a y'^2.$$

The sum of these results constitutes the total Resisting Moment, R.M., for the section.

Hence, $$\text{R.M.} = \frac{E}{\rho} \Sigma a y^2 + \frac{E}{\rho} \Sigma a y'^2.$$

There is now no longer any need for distinguishing between y and y', since the process of summation is the same all over the cross-section. We, therefore, finally get:—

$$\text{R.M.} = \frac{E}{\rho} \Sigma a y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III)}$$

The quantity $\Sigma a y^2$, being a purely geometrical function, depending only on the *form* of the section, is termed its

Moment of Inertia, and is usually denoted by the symbol I, and sometimes by the product $A k^2$ (see Lecture XXII). Table II., Lecture XXII., gives the values of k^2 for most of the sections required in the following examples. These multiplied by A will give the required values of I.

Writing I for $\Sigma a y^2$, our equations become:—

$$\text{B.M.} = \text{R.M.} = \frac{E I}{\rho};$$

$$\text{Or, the curvature, } \frac{1}{\rho} = \frac{M}{E I} \quad \ldots\ldots\ldots\ldots \quad \text{(IV)}$$

Where M stands for either the B.M. or R.M.

Again, from equations (I) and (I_a), we get:—

$$\left.\begin{aligned} \frac{f}{y} &= \frac{E}{\rho} = \frac{M}{I} \\ \therefore \qquad M &= \frac{f}{y} I \\ \text{Or,} \qquad f &= \frac{M}{I} y \end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(V)}$$

Formulæ (IV) and (V) are the fundamental equations of the theory of the strength of beams and girders. In applying the latter equation, it must always be borne in mind that f stands for either the tensile or compressive stress at any distance y, above or below the neutral axis.

The greatest stress comes on the fibres farthest from the neutral axis, and is the principal effect to be considered in questions of strength. If this is amply provided for, the beam will be safe. Let y now denote the distance of the fibres farthest from the neutral axis:—

$$\text{Then,} \qquad f_{max.} = \frac{M}{I} \times y = M \div \frac{I}{y}.$$

The ratio $\frac{I}{y}$ is usually denoted by Z, and is called the **Modulus of the Section.***

---

* See Prof. Unwin's *Elements of Machine Design*, Pages 56 to 59, for a table of the moduli of sections of beams, where $Z = \frac{I}{y}$. Also, in Seaton & Rounthwaite's *Pocket Book of Marine Engineering Rules and Tables*, from which we extracted the tables at the end of this lecture, we also find $Z = \frac{I}{y}$ as the modulus of the section.

Hence, writing Z for $\frac{I}{y}$, we have :—

$$\left.\begin{aligned} f_{max.} &= \frac{M}{Z} \\ \text{Or,}\qquad M &= Z f_{max.} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(VI)}$$

In applying this equation the student must be careful to remember that in those cases where the section of the beam or girder is not symmetrical about the neutral axis, there will be two values of y to be taken into account, and therefore two values of Z. On the whole, we think it safer to adhere to the general formula (V) as being less likely to lead to confusion; at the same time, it is very convenient to use equation (VI) in taking out quantities in the drawing office by aid of tables since it reduces the work of calculation.

EXAMPLE I.—A floor joist, 12 inches deep and 3 inches broad, has a span of 15 feet, and carries a uniformly distributed load of 1 cwt. per foot-run. Find the greatest intensity of stress within the timber. (S. and A. Adv. Exam., 1891.)

ANSWER.—In problems involving the calculation of stress within the beam, the student will find it best to express all dimensions in *inches*, and, therefore, bending moments in *inch-lbs.* or *inch-tons* as the case may be.

In this problem the greatest stresses will occur at the middle of the joist where the bending moment attains its maximum value, which, in this case, is :—

$$\text{Max. B.M.} = \tfrac{1}{8}\, w\, L^2 \text{ inch-lbs.}$$

$$\text{Here,}\qquad w = \frac{112}{12}\text{ lbs.}$$

$$\text{And,}\qquad L = 15 \times 12 \text{ inches.}$$

$$\therefore\qquad \text{B.M.} = \frac{1}{8} \times \left(\frac{112}{12}\right) \times (15 \times 12)^2 \text{ inch-lbs.}$$

$$\text{Or,}\qquad \text{B.M.} = 14 \times 15 \times 15 \times 12 \qquad ,,$$

The value of I for a rectangular section is :—

$$I = \frac{1}{12}(\text{breadth}) \times (\text{depth})^3,$$

$$\text{Or,}\qquad I = \frac{1}{12} \times 3 \times 12^3 = 3 \times 12 \times 12.$$

The greatest stress at the middle section of the joist will occur in the fibres farthest away from the neutral axis. Hence, $y = 6$ inches.

Applying equation (V) we have:—

$$f = \frac{\text{B.M.}}{\text{I}}\, y,$$

$$,, = \frac{14 \times 15 \times 15 \times 12}{3 \times 12 \times 12} \times 6 = \mathbf{525\ lbs.\ per\ sq.\ in.}$$

Example II.—A uniform beam of oak, 10 feet long, 15 inches deep and 10 inches wide, sustains, in addition to its own weight, a load of 5,000 lbs. placed at the centre. Find the greatest bending moment and the greatest stress in the fibres.

The specific gravity of oak is 0·934. (S. and A. Adv. Exam., 1894.)

Answer.—Here the greatest bending moment takes place at the centre of the beam and is made up of two parts: (1) that due to the beam's own weight which is uniformly distributed along its length; and (2) that due to the 5,000 lbs. concentrated at its middle.

For (1), $\text{B.M.}_1 = \frac{1}{8} w \text{L}^2$ inch-lbs.

And for (2), $\text{B.M.}_2 = \frac{1}{4} \text{W L}$,,

∴ Total, $\text{B.M.} = \frac{1}{8} w \text{L}^2 + \frac{1}{4} \text{W L}$ inch-lbs.

Taking the weight of a cubic inch of water as 0·036 lb., then a cubic inch of oak will weigh $0·934 \times 0·036 = 0·0336$ lb.

$$\therefore \quad w = 0·0336 \times 15 \times 10 = 5·04 \text{ lbs.}$$

$$\text{And,} \quad \text{B.M.} = \tfrac{1}{8} \times 5·04 \times (10 \times 12)^2 + \tfrac{1}{4} \times 5{,}000 \times (10 \times 12)$$

$$,, = 9{,}072 + 150{,}000 = \mathbf{159{,}072\ inch\text{-}lbs.}$$

$$\text{Here,} \quad \text{I} = \tfrac{1}{12} \times 10 \times 15^3 = \tfrac{1}{2} \times 5 \times 5 \times 15 \times 15$$

$$\text{And,} \quad y = \tfrac{1}{2} \times 15 \text{ inches.}$$

$$\therefore \quad f = \frac{\text{B.M.}}{\text{I}}\, y = \frac{159{,}072}{\frac{1}{2} \times 5 \times 5 \times 15 \times 15} \times \tfrac{1}{2} \times 15$$

$$,, = \mathbf{424·1\ lbs.\ per\ sq.\ inch.}$$

Example III.—A round steel spindle 10 inches long, and held at one end, revolves at the rate of 150 revolutions per minute round a vertical axis, to which the axis of the spindle is parallel and from which it is 2 feet distant. The spindle has a uniformly distributed load, the whole revolving weight being 30 lbs. What should be the diameter of the spindle when the safe working stress of the material in tension or compression is taken at 25,000 lbs. per square inch? (S. & A. Hons. Exam., 1891.)

ANSWER.—The spindle in this problem may be likened to a beam fixed at one end and carrying a uniformly distributed load. The load being not the revolving weight of 30 lbs., but the centrifugal force of that weight due to its being whirled round at the rate of 150 revolutions per minute.

$$\text{Velocity of spindle, } = \frac{150 \times 2\pi \times 2}{60} = 10\,\pi \text{ ft. per sec.}$$

$$\text{Centrifugal force, } = \frac{30 \times (10 \times \pi)^2}{32 \times 2} = \frac{1500 \times \pi^2}{32} \text{ lbs.}$$

This force, multiplied by half the length of the spindle, gives us the bending moment at the fixed end of the spindle :—

$$\text{That is, } \quad \text{B.M.} = \frac{1500 \times \pi^2}{32} \times \frac{10}{2} \text{ inch-lbs.}$$

If D be the diameter of the spindle in inches, then from Lecture XXII. we get :—

$$\text{The Moment of Inertia, I} = \frac{\pi}{64} D^4$$

$$\text{And the Modulus of Section, Z} = \frac{I}{\frac{D}{2}} = \frac{\pi}{32} D^3.$$

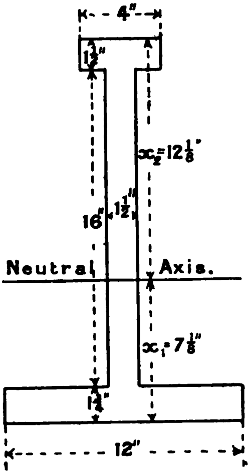
Now, fZ = B.M. ; and, f = 25,000 lbs. per sq. inch.

$$\therefore \quad 25{,}000 \times \frac{\pi}{32} D^3 = \frac{1500\,\pi^2}{32} \times \frac{10}{2}$$

$$\text{Or,} \quad D^3 = 0{\cdot}3 \times \pi = 0{\cdot}94248$$

$$\text{Hence,} \quad D = \sqrt[3]{0{\cdot}94248} = 0{\cdot}98 \text{ inch.}$$

EXAMPLE IV.—The section of a cast-iron girder, and the maximum safe tensile and compressive stresses being given, explain how to determine its moment of resistance to bending. The dimensions of the section of a cast-iron girder are the following:—Top flange, 4 by 1½ inches; bottom flange, 12 by 1¾ inches; web, 16 by 1½ inches. Determine the moment of resistance, the greatest permissible tensile and compressive stresses being 2½ and 7½ tons per square inch respectively. If the girder be 20 feet long, and is supported at its two ends, find the greatest safe load which it will carry when uniformly distributed along its length. (S. and A. Hons. Exam., 1895.)



SECTION OF GIRDER.

Answer.—As this is an excellent example for showing the student how the R.M. of a girder section is calculated, we shall go into the matter in detail. Let the accompanying figure represent the cross-section in question.

We have first to find the position of the neutral axis N A, by writing down the sectional areas of the parts composing the figure, and taking moments about the lower edge.

$$\text{Area of top flange} = 4 \times 1\tfrac{1}{2} = 6 \text{ sq. in.}$$

$$\text{,, bottom ,,} = 12 \times 1\tfrac{3}{4} = 21 \text{ ,,}$$

$$\text{,, web} = 16 \times 1\tfrac{1}{2} = 24 \text{ ,,}$$

$$\therefore \quad \text{Total area of section} = 51 \text{ ,,}$$

Then, since N A passes through the centre of area of the section, we have:—

$$51 \times x_1 = 6 \times 18\tfrac{1}{2} + 24 \times 9\tfrac{3}{4} + 21 \times \tfrac{7}{8} = 363\tfrac{3}{8}.$$

$$\therefore \quad x_1 = \frac{363\tfrac{3}{8}}{51} = 7\tfrac{1}{8}.$$

$$\text{And} \quad x_2 = 19\tfrac{1}{4} - 7\tfrac{1}{8} = 12\tfrac{1}{8}.$$

We calculate the value of I, the moment of inertia of the section about the neutral axis, by finding that for each of the parts into which the section is divided and taking their sum. As the neutral axis does not pass through the centre of any of those parts, we shall have to employ Proposition I. of Lecture XXII., to which we again refer the student.

Remembering that the moment of inertia of a rectangular area about an axis through its centre of gravity is :—

$\frac{1}{12}$ (breadth) × (depth)^3, we have :—

$$\text{For top flange, } I_t = \tfrac{1}{12} \times 4 \times (1\tfrac{1}{2})^3 + 6 \times (11\tfrac{3}{8})^2.$$

$$\text{,, ,, ,, } = 1{\cdot}125 + 776{\cdot}343 = 777{\cdot}468.$$

$$\text{For bottom flange, } I_b = \tfrac{1}{12} \times 12 \times (1\tfrac{3}{4})^3 + 21 \times (6\tfrac{1}{4})^2.$$

$$\text{,, ,, ,, } = 5{\cdot}359 + 820{\cdot}312 = 825{\cdot}671.$$

$$\text{For web, } I_w = \tfrac{1}{12} \times 1\tfrac{1}{2} \times (16)^3 + 24 \times (2\tfrac{5}{8})^2.$$

$$\text{,, ,, } = 512{\cdot}0 + 165{\cdot}375 = 677{\cdot}375.$$

$$\therefore \text{ For whole section, } I = 777{\cdot}468 + 825{\cdot}671 + 677{\cdot}375 = 2280{\cdot}5.^*$$

To illustrate what we said about the moduli of unsymmetrical sections, we shall find both moduli for this example :—

$$\text{For tension, } Z_t = \frac{I}{x_1} = \frac{2280{\cdot}5}{7{\cdot}125} = 320{\cdot}0.$$

$$\text{For compression, } Z_c = \frac{I}{x_2} = \frac{2280{\cdot}5}{12{\cdot}125} = 188{\cdot}0.$$

The question gives as the greatest permissible values for :—

Tensile stress, f_t = 2·5 tons per sqr. inch.

Compressive stress, f_c = 7·5 ,, ,,

Since, R.M. = $Zf_{max.}$, we must take the lower of the two values of R.M. in fixing the load to be carried by the girder. These are :—

$$Z_t \times f_t = 320 \times 2{\cdot}5 = 800 \text{ inch-tons.}$$

$$\text{And, } Z_c \times f_c = 188 \times 7{\cdot}5 = 1410 \text{ inch-tons.}$$

$$\therefore \text{ B.M.} = \text{R.M.} = 800 \text{ inch-tons.}$$

---

* Another and rather shorter method of finding I for this form of section is to (1) produce the sides of top and bottom flanges to meet the neutral axis N A, (2) calculate the moments of inertia of the two full rectangles thus formed, (3) subtract from their sum the moments of inertia of the four rectangular areas which are in excess of the section of the beam. All these moments may be found by the formula $I = \frac{1}{12} B \times D^3$, which will only require to be used four times as the blank rectangles on each side of the web are equal in pairs.

The girder will, therefore, safely carry a uniformly distributed load, given by the equation :—

$$\tfrac{1}{8}\, w\, L^2 = 800.$$

$$\therefore \qquad W = w\, L = \frac{8 \times 800}{20 \times 12} = 26\tfrac{2}{3} \text{ tons.}$$

This will make the maximum compressive stress

$$f_{c\,max.} = \frac{800}{188} = 4{\cdot}255 \text{ tons,}$$

instead of 7·5 as given; showing that the girder is not well designed.

In a properly proportioned section we should have :—

$$Z_t \times f_t = Z_c \times f_c.$$

Thin Wrought-Iron Girders.—In the case of wrought-iron girders where the flanges are thin compared with their distance apart, and where the bending resistance of the web is disregarded as a provision against the shearing force acting at the section, the formulæ for the moment of resistance are very simple.

Let A_t = Area of flange in tension.
" A_c = Area of flange in compression.
" H = Distance between centres of flanges.
" f_t = Mean stress in tension flange.
" f_c = Mean stress in compression flange.

Distance between centre of tension flange and the neutral axis is :—

$$y_t = \left(\frac{A_c}{A_t + A_c}\right) H;$$

and the distance between centre of compression flange and the neutral axis is :—

$$y_c = \left(\frac{A}{A_t + A_c}\right) H.$$

The moment of inertia of the flanges, with respect to the neutral axis, is :—

$$I = A_t \left(\frac{A_c}{A_t + A_c}\right)^2 H^2 + A_c \left(\frac{A_t}{A_t + A_c}\right)^2 H^2$$

Or, $$\mathrm{I} = \{\mathrm{A}_t \times \mathrm{A}_c^2 + \mathrm{A}_c \times \mathrm{A}_t^2\} \times \left(\frac{\mathrm{H}}{\mathrm{A}_t + \mathrm{A}_c}\right)^2$$

Since, $f = \dfrac{\mathrm{M}}{\mathrm{I}}\, y$.

$$\therefore \quad f_t = \frac{\mathrm{M}}{\{\mathrm{A}_t \times \mathrm{A}_c^2 + \mathrm{A}_c \times \mathrm{A}_t^2\}\left(\dfrac{\mathrm{H}}{\mathrm{A}_t + \mathrm{A}_c}\right)^2} \times \left(\frac{\mathrm{A}_c}{\mathrm{A}_t + \mathrm{A}_c}\right)\mathrm{H}$$

$$\left.\begin{aligned} &\text{Hence,} && f_t = \frac{\mathrm{M}}{\mathrm{A}_t \times \mathrm{H}} \\ &\text{Similarly,} && f_c = \frac{\mathrm{M}}{\mathrm{A}_c \times \mathrm{H}} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(VII)}$$

Example V.—A wrought-iron riveted girder of I section has a top flange of 9 square inches in sectional area, and a bottom flange of 8 square inches. The distance between the centres of gravity of the flanges is 12 inches, and the ends of the beam rest on abutments, 16 feet apart. The girder being loaded uniformly with a load equal to 1 ton per lineal foot (including the weight of the beam). What would be the mean stress per square inch on the metal in each flange at the dangerous section? The resistance of the web to bending is neglected. (S. & A. Hons. Exam., 1892.)

Answer.—By "dangerous section" is here meant the middle section of the girder, where the maximum bending moment occurs. (See equation (XI_a) of Lecture XXXI.)

$$\text{Max. B.M.} = \frac{1}{8}\left(\frac{1}{12}\right) \times (16 \times 12)^2 = 32 \times 12 \text{ inch-tons.}$$

Hence, mean stress in tension flange,

$$f_t = \frac{32 \times 12}{8 \times 12} = 4 \text{ tons per square inch.}$$

And, mean stress in compression flange,

$$f_c = \frac{32 \times 12}{9 \times 12} = 3{\cdot}55 \text{ tons per square inch.}$$

Curvature and Deflection of Beams.—When we speak of the curvature or the deflection of a beam we mean that of its neutral surface.

If the beam is fixed at one end, we take the origin of co-ordinates at that end; but if supported or fixed symmetrically at both ends, we take it at the middle.

Let the co-ordinates of the neutral surface curve be denoted, as usual, by x and y, then the deflection of the beam at any distance x, from the origin will be measured by y and the tangent of its inclination to the horizon by $\frac{dy}{dx}$.

The equation of the curve into which the beam is bent will be:—

$$y = \varphi(x),$$

Where $\varphi(x)$ is a function of x to be determined for each particular case.

In treatises on the analytical geometry of plane curves it is shown that the general expression for radius of curvature is:—

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Although of great importance, in the theory of beams $\frac{dy}{dx}$ is always such a small fraction, that its square becomes a perfectly negligible quantity in comparison with unity. We may, therefore, safely disregard the value of $\left(\frac{dy}{dx}\right)^2$ in the above formula, and write $\frac{d^2y}{dx^2} = \frac{1}{\rho}$.

But by equation (IV) of this Lecture we know that:—

$$\frac{1}{\rho} = \frac{M}{EI}. \quad \therefore \frac{d^2y}{dx^2} = \frac{M}{EI}. \quad . \quad . \quad . \quad . \quad \text{(VIII)}$$

In what follows we shall assume that the beam or girder is of uniform section so that I is constant, the more general cases where I varies being rather beyond the scope of this treatise.

We shall begin by working out the following example, which will form a good introduction to this rather mathematical part of our subject.

Example VI.—Investigate a formula for calculating the amount of deflection of a beam supported at its ends and loaded uniformly. Find the deflection in a beam of timber of uniform rectangular section, 6 inches wide and 12 inches deep, the beam being supported at its ends in a horizontal position on two walls 12 feet apart. There is to be taken into account a single concentrated load of 4,000 lbs. at the centre, and a uniformly distributed load of 2,500 lbs., the modulus of elasticity being 1,750,000 lbs. per square inch. (S. & A. Hons. Exam., 1891.)

ANSWER.—Taking the middle of the beam as the origin of co-ordinates, we have already proved (see equation (XI) in Lecture XXXI.) that the bending moment at x inches from this point, in the case of a beam L inches between supports, and loaded uniformly with w lbs. per inch-run, is:—

$$\text{B.M.} = \tfrac{1}{2} w \left(\tfrac{1}{4} L^2 - x^2\right) \text{ inch-lbs.}$$

Substituting this in formula (VIII) we get:—

$$\frac{d^2 y}{dx^2} = \frac{w}{2 E I} \left(\tfrac{1}{4} L^2 - x^2\right).$$

Now, multiplying both sides by dx, and integrating, we have:—

$$\frac{dy}{dx} = \frac{w}{2 E I} \int \left(\tfrac{1}{4} L^2 - x^2\right) dx.$$

Or,
$$\frac{dy}{dx} = \frac{w}{2 E I} \left(\tfrac{1}{4} L^2 x - \tfrac{1}{3} x^3\right).$$

This needs no correction because $\dfrac{dy}{dx} = 0$, when $x = 0$.

Integrating a second time, we get:—

$$y = \frac{w}{2 E I} \int \left(\tfrac{1}{4} L^2 x - \tfrac{1}{3} x^3\right) dx.$$

Or,
$$y = \frac{w}{8 E I} \left(\tfrac{1}{2} L^2 x^2 - \tfrac{1}{3} x^4\right). \quad . \quad . \quad . \quad . \quad . \quad (\text{IX})$$

This also requires no correction, as x and y vanish together. Now, let Δ_1 denote the deflection of the beam for the distributed load:—

$$\Delta_1 = y, \text{ when } x = \tfrac{1}{2} L.$$

Or,
$$\Delta_1 = \frac{w}{8 E I} \left\{\tfrac{1}{2} L^2 \left(\tfrac{1}{2} L\right)^2 - \tfrac{1}{3} \left(\tfrac{1}{2} L\right)^4\right\}.$$

$$\therefore \quad \Delta_1 = \frac{5 w L^4}{384 E I} \text{ inches.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{X})$$

In the case of a beam carrying a single load of W lbs. at its middle point, the bending moment due to that load at x inches from the middle point (Equation (VII) Lecture XXXI.) is :—

$$\text{B.M.} = \tfrac{1}{2} W \left(\tfrac{1}{2} L - x\right) \text{ inch-lbs.}$$

$$\therefore \qquad \frac{d^2 y}{d x^2} = \frac{W}{2 E I} \left(\tfrac{1}{2} L - x\right).$$

The first integration of this equation gives :—

$$\frac{d y}{d x} = \frac{W}{2 E I} \left(\tfrac{1}{2} L x - \tfrac{1}{2} x^2\right) = \frac{W}{4 E I} (L x - x^2).$$

And the second integration :—

$$y = \frac{W}{4 E I} \left(\tfrac{1}{2} L x^2 - \tfrac{1}{3} x^3\right). \quad . \quad . \quad . \quad . \quad . \quad (\text{XI})$$

If Δ_2 be the total deflection in this case, then Δ_2 is the value of y when $x = \frac{1}{2} L$.

$$\therefore \qquad \Delta_2 = \frac{W}{4 E I} \left\{\tfrac{1}{2} L \left(\tfrac{1}{2} L\right)^2 - \tfrac{1}{3} \left(\tfrac{1}{2} L\right)^3\right\}.$$

$$\text{Or,} \qquad \Delta_2 = \frac{W L^3}{48 E I} \text{ inches.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XII})$$

If the beam carry both loads at the same time, as given in the question, then the total deflection due to the two loads will be :—

$$\Delta = \Delta_1 + \Delta_2.$$

$$\therefore \qquad \Delta = \frac{5 w L^4}{384 E I} + \frac{W L^3}{48 E I}.$$

$$\text{Or,} \qquad \Delta = \frac{L^3}{48 E I} \left(\tfrac{5}{8} w L + W\right).$$

The numerical data given are :—

$$L = 12 \times 12 \text{ inches.}$$

$$I = \tfrac{1}{12} b d^3 = \tfrac{1}{12} \times 6 \times 12^3 = 6 \times 12^2.$$

$$W = 4{,}000 \text{ lbs.}$$

$$w L = 2{,}500.$$

$$\text{And,} \qquad E = 1{,}750{,}000.$$

$$\therefore \qquad \Delta = \frac{(12 \times 12)^3}{48 \times 1{,}750{,}000 \times 6 \times 12^2} \left(\tfrac{5}{8} \times 2{,}500 + 4{,}000\right)$$

$$\Delta = 0{\cdot}2288 \text{ inches.}$$

Uniform Beam on Three Supports.—Suppose we are given a uniform beam resting on three supports all on the same level, to find the pressure on the middle support.

It is clear that if the middle support were taken away, the weight of the beam would cause it to bend down at the middle [as found above by equation (X)] through a distance

$$\Delta_1 = \frac{5\,w\,L^4}{384\,E\,I} \text{ inches.}$$

We have also seen by equation (XII) that a single concentrated load of W lbs. applied at the middle of the beam would produce an amount of deflection, $\Delta_2 = \frac{W\,L^3}{48\,E\,I}$ inches.

This gives us the upward deflection caused by the reaction of the central support if we put its value, P, instead of W in the equation.

The total deflection will be zero if all three supports are on the same level.

Then, $$\frac{P\,L^3}{48\,E\,I} = \frac{5\,w\,L^4}{384\,E\,I}.$$

Or, $$P = \tfrac{5}{8}\,w\,L.$$

The pressure on the middle support is thus seen to be $\frac{5}{8}$ of the weight of the beam; whilst the end supports each carry $\frac{3}{16}$ of the weight.

Uniform Beam Fixed at one End and Supported at the other.—If w be the weight of the beam in lbs. per inch-run and L its length in inches, then, we already know that at x inches from the fixed end, the

$$\text{B.M.} = \tfrac{1}{2}\,w\,(L - x)^2 \text{ inch-lbs.}$$

Putting this value of the B.M. in equation (VIII), we get:—

$$\frac{d^2 y}{d x^2} = \frac{w}{2\,E\,I}(L - x)^2 = \frac{w}{2\,E\,I}(L^2 - 2\,L\,x + x^2).$$

$$\therefore \quad \frac{d\,y}{d\,x} = \frac{w}{2\,E\,I}\int(L^2 - 2\,L\,x + x^2)\,d\,x$$

$$,, \quad = \frac{w}{2\,E\,I}\{L^2 x - L\,x^2 + \tfrac{1}{3}\,x^3\}.$$

And, $$y = \frac{w}{2\,E\,I}\int\{L^2 x - L\,x^2 + \tfrac{1}{3}\,x^3\}\,d\,x$$

$$y = \frac{w}{2\,E\,I}\{\tfrac{1}{2}\,L^2 x^2 - \tfrac{1}{3}\,L\,x^3 + \tfrac{1}{12}\,x^4\}. \quad . \quad . \quad \text{(XIII)}$$

This last equation gives the droop of the beam at any distance x, from the fixed end. At the free end let Δ_1 be the value of y when $x = L$.

Then,
$$\Delta_1 = \frac{w L^4}{8 E I} \text{ inches. . . . . . } \quad \text{(XIV)}$$

Let P be the upward pressure in lbs., between the beam and a support placed under its free end. The bending moment, due to P, at x inches from the fixed end is $P(L - x)$ inch-lbs. Hence, the curvature produced by P will be:—

$$\frac{d^2 y}{d x^2} = \frac{P}{E I}(L - x).$$

$$\therefore \quad \frac{dy}{dx} = \frac{P}{EI}\int(L - x)\,dx = \frac{P}{EI}(Lx - \tfrac{1}{2}x^2).$$

And,
$$y = \frac{P}{EI}\int(Lx - \tfrac{1}{2}x^2)\,dx = \frac{P}{2EI}(Lx^2 - \tfrac{1}{3}x^3). \quad . \quad \text{(XV)}$$

When $x = L$, let $y = \Delta_2$.

$$\therefore \quad \Delta_2 = \frac{P L^3}{3 E I} \text{ inches. . . . . . } \quad \text{(XVI)}$$

If $\Delta_2 = \Delta_1$, the supported end will be raised to the same level as the fixed end.

Then,
$$\frac{P L^3}{3 E I} = \frac{w L^4}{8 E I}.$$

Or,
$$P = \tfrac{3}{8} w L \text{ lbs.}$$

This result shows that the pressure on the prop is equal to $\frac{3}{8}$ of the weight of the beam.

It will be instructive for the student to observe that this result might easily have been inferred from the previous case of a beam resting on three props.

In that case the part of the beam immediately over the middle support is in exactly the same condition as the fixed end of the beam in this case; so that whatever is true of each half of the beam in the former case will here hold good for the whole beam. The pressure on the end supports is, therefore, identical in magnitude in each case; because $\frac{3}{16}$ of the weight of the whole beam is the same thing as $\frac{3}{8}$ of the weight of each half.

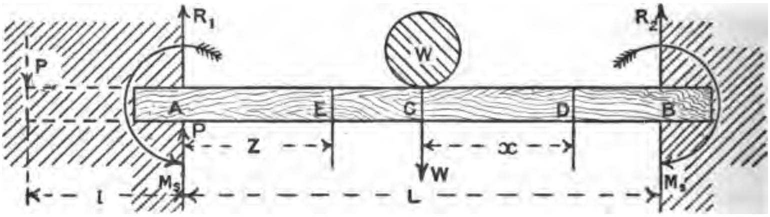
Beam Fixed at Both Ends and Loaded at the Centre.—When a beam is fixed, or built horizontally into a wall at both ends, the fixing causes a bending moment which is constant all over the beam. For the reaction of the left support in keeping the beam horizontal is equivalent to a force P, acting downwards at some distance l, to the left of that support, and an upward force P, at the support. The bending moment at the support is then :—

$$M_s = P \times l.$$

And, at any other point, E, of the beam, at a distance, z (less than half the span from the support), the B.M. caused by this reaction at the support is :—

$$\text{B.M.} = P(z + l) - Pz = Pl = M_s.$$

Consequently, the fixing at the ends causes a constant B.M. all over the beam, equal to that at the supports, in addition to that caused by the load (*but in the opposite direction*).



Beam Fixed at Ends and Loaded at Centre.

Taking our origin of co-ordinates at C, the centre, and the undeflected axis, or neutral line of the beam, as our axis of x, we have, at a section D, distant x from C :—

$$\text{B.M.} = R_2(\tfrac{1}{2}L - x) - M_s = \tfrac{1}{2}W(\tfrac{1}{2}L - x) - M_s$$

Hence, from eqn. (VIII), $$\frac{d^2y}{dx^2} = \frac{1}{EI}\{\tfrac{1}{2}W(\tfrac{1}{2}L - x) - M_s\}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{EI}\{\tfrac{1}{4}W(Lx - x^2) - M_s x\}$$

The beam is horizontal at the centre and at the ends, therefore $\frac{dy}{dx}$ is zero when x is zero, and when $x = \tfrac{1}{2}L$.

$$\therefore \quad 0 = \frac{1}{EI}\{\tfrac{1}{4}W(\tfrac{1}{2}L^2 - \tfrac{1}{4}L^2) - \tfrac{1}{2}M_s L\}$$

Or, $$M_s = \tfrac{1}{8}WL.$$

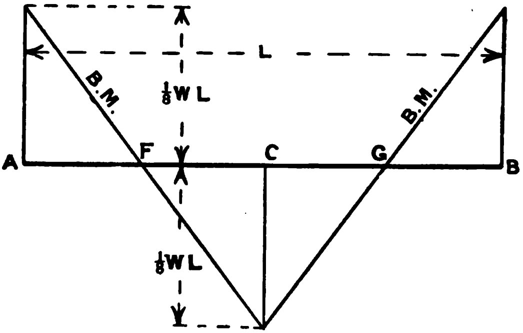
Inserting this value in the above equation for the B.M. we get :—

$$\text{B.M.} = \tfrac{1}{2}\text{W}\left(\tfrac{1}{2}\text{L} - x\right) - \tfrac{1}{8}\text{W L}$$

Or, $$\text{B.M.} = \tfrac{1}{2}\text{W}\left(\tfrac{1}{4}\text{L} - x\right) \quad . \quad . \quad . \quad . \quad \text{(XVII)}$$

At the centre, $$\text{B.M.} = \tfrac{1}{2}\text{W} \,.\, \tfrac{1}{4}\text{L} = \tfrac{1}{8}\text{W L} = \text{M}_8$$

$$\therefore \quad \text{Maximum B.M.} = \text{M}_8 = \tfrac{1}{8}\text{W L.} \quad . \quad . \quad . \quad . \quad . \quad \text{(XVIII)}$$



B.M. DIAGRAM FOR BEAM FIXED AT ENDS AND LOADED AT CENTRE.

We thus see that, in this case the fixing of the ends reduces the maximum B.M. to half what it would be with free ends, and that this maximum B.M. occurs both at the centre and the ends.

The B.M. diagram is similar to what we had for a beam simply supported, but the base line is shifted half way down the diagram, so that it is crossed at F and G by the lines representing the B.M. It will be seen from equation (XVII) that the B.M. is zero where $x = \frac{1}{4}$ L, and that it is positive on one side of this point, and negative on the other. This is one of the points where the B.M. curve cuts the base line, and it is called a *point of inflection*, because the beam is straight just at that point and the curvature changes sign. There is, of course, another point of inflection at the distance $\frac{1}{4}$ L on the other side of the centre.

In large girder bridges that part of the span between the two points of inflection is made separate from the remainder and rests on rollers at these points. This allows freedom of expansion without reducing the strength of the bridge.

Integrating the value of $\dfrac{dy}{dx}$ we get :—

$$y = \frac{1}{\text{E I}}\left\{\tfrac{1}{4}\text{W}\left(\tfrac{1}{2}\text{L}x^2 - \tfrac{1}{3}x^3\right) - \tfrac{1}{16}\text{W L}x^2\right\}.$$

Therefore, at the ends, where $x = \frac{1}{2}L$:—

$$y = \frac{1}{EI}\left\{\frac{1}{4}W\left(\frac{L^3}{8} - \frac{L^3}{24}\right) - \frac{1}{16}W\frac{L^3}{4}\right\} = \frac{WL^3}{192\,EI}.$$

Hence, the difference of level between the centre and the ends is :—

$$\Delta = \frac{WL^3}{192\,EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (XIX)$$

This is only one-fourth of the deflection when the beam simply rested on its supports (Equation XII), so that the beam is now four times as stiff.

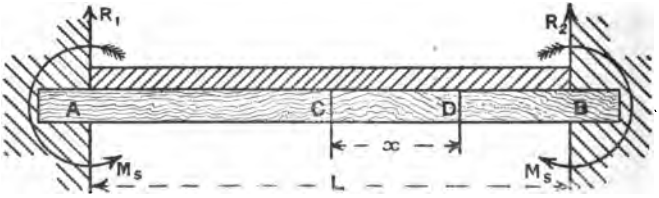
Beam Fixed at Both Ends and Loaded Uniformly.—Taking axes as before, the B.M. at any section, D, is :—

$$\text{B.M.} = R_2\left(\tfrac{1}{2}L - x\right) - \tfrac{1}{2}w\left(\tfrac{1}{2}L - x\right)^2 - M_s.$$

$$\text{Or, B.M.} = \tfrac{1}{2}wL\left(\tfrac{1}{2}L - x\right) - \tfrac{1}{2}w\left(\tfrac{1}{4}L^2 - Lx + x^2\right) - M_s.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{EI}\left\{\tfrac{1}{2}w\left(\tfrac{1}{4}L^2 - x^2\right) - M_s\right\}.$$

$$\text{Or,} \quad \frac{dy}{dx} = \frac{1}{EI}\left\{\tfrac{1}{2}w\left(\tfrac{1}{4}L^2x - \tfrac{1}{3}x^3\right) - M_sx\right\}.$$



BEAM FIXED AT BOTH ENDS AND LOADED UNIFORMLY.

In this case also, $\frac{dy}{dx}$ is zero when x is zero, and when $x = \frac{1}{2}L$.

$$\therefore \quad 0 = \tfrac{1}{2}wL^3\left(\tfrac{1}{8} - \tfrac{1}{24}\right) - \tfrac{1}{2}M_sL.$$

$$\text{Or,} \quad M_s = \frac{wL^2}{12} = \frac{WL}{12}.$$

$$\text{Hence,} \quad \text{B.M.} = \tfrac{1}{2}w\left(\tfrac{1}{4}L^2 - x^2\right) - \tfrac{1}{12}wL^2.$$

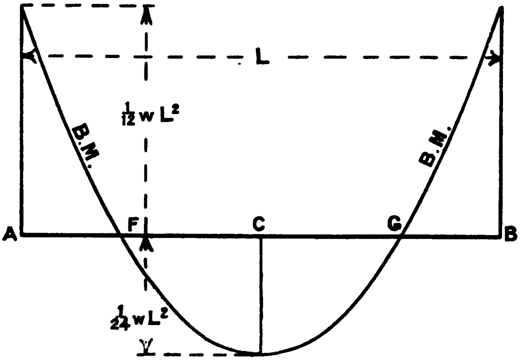
$$\text{Or,} \quad \text{B.M.} = \tfrac{1}{2}w\left(\tfrac{1}{12}L^2 - x^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (XX)$$

At the centre, where $x = 0$, the B.M. is :—

$$M_c = \frac{wL^2}{24} = \frac{WL}{24} = \tfrac{1}{2}M_s.$$

This is only half of that at the support. Hence, the greatest bending moment occurs at the support, and its value is:—

$$\text{Maximum B.M.} = \tfrac{1}{12}\,w\,L^2 = \tfrac{1}{12}\,W\,L. \quad . \quad . \quad \text{(XXI)}$$



B.M. Diagram for Beam Fixed at both Ends and Loaded Uniformly.

The points of inflection occur where $x^2 = \frac{1}{12}\,L^2$ or $x = \pm\frac{L}{2\sqrt{3}}$.

By integrating the above value of $\frac{d\,y}{d\,x}$:—

$$y = \frac{1}{\mathrm{EI}}\left\{\tfrac{1}{2}\,w\,(\tfrac{1}{8}\,L^2 x^2 - \tfrac{1}{12}\,x^4) - \tfrac{1}{12}\,w\,L^2\frac{x^2}{2}\right\} = \frac{w}{48\,\mathrm{E\,I}}(L^2 x^2 - 2\,x^4).$$

Putting $x = \frac{L}{2}$ we obtain the amount by which the centre of the beam is deflected by the load, viz.:—

$$\Delta = \frac{w}{48\,\mathrm{E\,I}}\left(\frac{L^4}{4} - \frac{L^4}{8}\right) = \frac{w\,L^4}{384\,\mathrm{E\,I}} = \frac{W\,L^3}{384\,\mathrm{E\,I}} \quad . \quad . \quad \text{(XXII)}$$

We thus see that, by fixing the ends horizontally for this manner of loading, the strength of the beam is increased in the ratio 3 : 2, and its stiffness in the ratio 5 : 1.

When the span of the beam is small, it may be designed wholly from considerations of strength; but when the span is great a beam may be strong enough, and yet not suitable, because it yields too much when the load is put on it. It then becomes necessary to take the stiffness into account by using one of the formulæ we have found for the deflection. The greatest deflection usually allowed in beams is 1 inch in 100 feet, or $\frac{1}{1200}$ of the span.

In the tables below we give a summary of these results, showing the relation between them.

TABLE III.—STRENGTH AND STIFFNESS OF BEAMS UNDER A TOTAL LOAD OF W LBS.

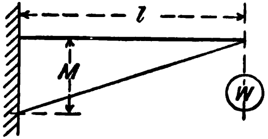
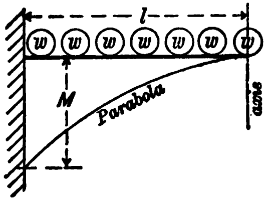
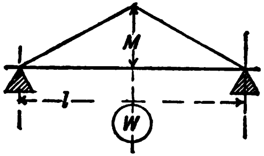
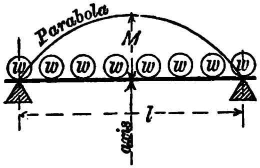
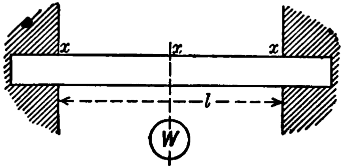
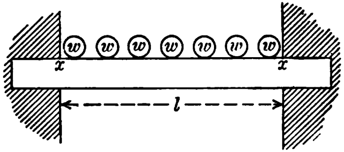
No.	MANNER OF SUPPORTING AND LOADING.	Maximum Bending Moment $= M$.	Relative Strength.	Deflection in Terms of W.	Deflection in Terms of M.	Deflection in Terms of Stress.	Relative Stiffness under same Load.
I.	 CANTILEVER LOADED AT END.	$W l$	$\frac{1}{4}$	$\frac{1}{3} \cdot \frac{W l^3}{E I}$	$\frac{1}{3} \cdot \frac{M l^2}{E I}$	$\frac{1}{3} \cdot \frac{f l^2}{E y}$	$\frac{1}{16}$
II.	 l w w w w w w w M Parabola axis CANTILEVER LOADED UNIFORMLY.	$\frac{W l}{2}$	$\frac{1}{2}$	$\frac{1}{8} \cdot \frac{W l^3}{E I}$	$\frac{1}{4} \cdot \frac{M l^2}{E I}$	$\frac{1}{4} \cdot \frac{f l^2}{E y}$	$\frac{1}{6}$
III.	 SUPPORTED AT BOTH ENDS. LOADED AT CENTRE.	$\frac{W l}{4}$	1	$\frac{1}{48} \cdot \frac{W l^3}{E I}$	$\frac{1}{12} \cdot \frac{M l^2}{E I}$	$\frac{1}{12} \cdot \frac{f l^2}{E y}$	1

TABLE III.—STRENGTH AND STIFFNESS OF BEAMS UNDER A TOTAL LOAD OF W LBS. (*continued*).

No.	MANNER OF SUPPORTING AND LOADING.	Maximum Bending Moment = M	Relative Strength.	Deflection in Terms of W.	Deflection in Terms of M.	Deflection in Terms of Stress.	Relative Stiffness under same Load.
IV.	 SUPPORTED AT BOTH ENDS. LOADED UNIFORMLY.	$\frac{Wl}{8}$	2	$\frac{5}{384} \cdot \frac{Wl^3}{EI}$	$\frac{5}{48} \cdot \frac{Ml^2}{EI}$	$\frac{5}{48} \cdot \frac{fl^2}{Ey}$	$\frac{8}{5}$
V.	 FIXED AT ENDS. LOADED AT CENTRE.	$\frac{Wl}{8}$	2	$\frac{1}{192} \cdot \frac{Wl^3}{EI}$	$\frac{1}{24} \cdot \frac{Ml^2}{EI}$	$\frac{1}{24} \cdot \frac{fl^2}{Ey}$	4
VI.	 FIXED AT ENDS. LOADED UNIFORMLY.	$\frac{Wl}{12}$	3	$\frac{1}{384} \cdot \frac{Wl^3}{EI}$	$\frac{1}{32} \cdot \frac{Ml^2}{EI}$	$\frac{1}{32} \cdot \frac{fl^2}{Ey}$	8

The quantities in the sixth column are obtained by substituting the value of the maximum B.M. given by the third column in the fifth, and for those in the seventh we have put the value of M $\left(\text{viz., } \frac{fI}{y}\right)$ found in equation (V).

We also print for reference* a table of the strengths of materials and of the moduli of different sections.

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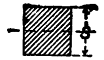
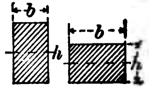
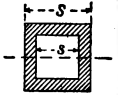
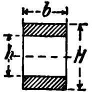
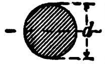
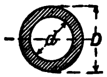
* From Seaton & Rounthwaite's *Pocket Book of Marine Engineering Rules and Tables*, which may be consulted for other cases of beams; or Unwin's *Machine Design*, Part I.

STRENGTHS, &c., OF MATERIALS (SUMMARY).

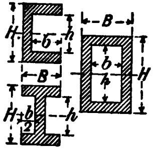
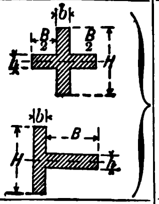
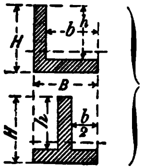
Material.	Ultimate Tensile Strength. lbs. per square inch.	Elastic Strength. lbs. per square inch.	Elongation per cent., when broken by Tensile Stress.
Cast-iron (ordinary good)	18,000	11,000	...
„ (Admiralty), .	not less than 20,160	...	...
Wrought-iron bars (ordinary good), . .	54,000	29,000	15 °/o in 8 ins.
Yorkshire plate—			
With grain, . .	54,000	26,000	20 °/o „
Across „ . .	49,000	...	14 °/o „
Staffordshire plate—			
With grain, . .	50,000	24,000	12 °/o „
Across „ . .	41,000	...	8 °/o „
Iron forgings—			
Large, . . .	45,000	...	9 °/o „
Small, . . .	50,000	...	13 °/o „
Steel castings (ordinary good), . . .	67,000	35,000	10 °/o „
Steel castings (Admiralty)	not less than 63,000	...	not less than 13½-18½ °/o in 2 ins.
„ (Lloyd's), .	not exceeding 67,000	...	not less than 10 °/o in 8 ins.
Steel boiler plate—			
(Ordinary good), .	65,000	36,000	20 °/o „
(Admiralty) internal,	not exceeding 60,480	...	20 °/o „
„ shell, .	60,480-67,200	not less than 31,360	...
(B. of T.) internal, .	58,240-67,200	...	...
„ shell, .	60,480-71,680	...	18 °/o in 10 ins.
Lloyd's, . . .	58,240-67,200	...	not less than 20 °/o in 8 ins.
Steel forgings (Admiralty)	62,720-78,400	34,500 to 43,120	28 °/o to 24 °/o in 2 ins.
Sheet copper, . .	30,000	5,600	35 °/o in 8 ins.
Copper wire (annealed),	40,000	...	...
Gun-metal (ordinary good). . . .	27,000	6,500	10 °/o in 2 ins.
Gun-metal (Admiralty),	31,000	...	...
Phosphor bronze (cast),	35,000	19,000	12 °/o in 2 ins.
Manganese bronze „ .	55,000	...	10 °/o „
„ (rolled),	67,000	...	20 °/o „
Muntz metal, . .	50,000	30,000	30 °/o „
Naval brass, . . .	54,000	24,000	25 °/o in 8 ins.

MOMENT OF INERTIA, MODULUS, &c., OF SOME SECTIONS.

The plane of bending is supposed perpendicular to plane of paper, and parallel to side of page.

Form of Section.	Area of Section. A	Moment of Inertia of Section about Axis through Centre of Gravity. I	Square of rads. of gyration of Section. $\rho = \frac{I}{A}$	"Modulus" of Section. $Z = \frac{I}{y}$
	s^2	$\frac{s^4}{12}$	$\frac{s^2}{12}$	$\frac{s^3}{6}$
	bh	$\frac{bh^3}{12}$	$\frac{h^2}{12}$	$\frac{bh^2}{6}$
	$S^2 - s^2$	$\frac{S^4 - s^4}{12}$	$\frac{S^2 + s^2}{12}$	$\frac{1}{6}\left(\frac{S^4 - s^4}{S}\right)$
	$b(H - h)$	$\frac{b(H^3 - h^3)}{12}$	$\frac{H^2 + hH + h^2}{12}$	$\frac{b(H^3 - h^3)}{6H}$
	$\cdot 7854 d^2$	$\cdot 0491 d^4$	$\frac{d^2}{16}$	$\cdot 0982 d^3$
	$\cdot 7854 (D^2 - d^2)$	$\cdot 0491 (D^4 - d^4)$	$\frac{D^2 + d^2}{16}$	$\cdot 0982 \left(\frac{D^4 - d^4}{D}\right)$

MOMENT OF INERTIA, MODULUS, &c., OF SOME SECTIONS—*Continued.*

Form of Section.	Area of Section. A	Moment of Inertia of Section about Axis through Centre of Gravity. I	Square of rads. of gyration of Section. $\rho^2 = \frac{I}{A}$	"Modulus" of Section. $Z = \frac{I}{y}$
	$BH - bh$	$\frac{BH^3 - bh^3}{12}$	$\frac{1}{12}\left(\frac{BH^3 - bh^3}{BH - bh}\right)$	$\frac{BH^3 - bh^3}{6H}$
	$Bh + bH$	$\frac{Bh^3 + bH^3}{12}$	...	$\frac{Bh^3 + bH^3}{6H}$
	$BH - bh$	$\frac{(BH^2 - bh^2)^2 - 4BHbh(H - h)^2}{12(BH - bh)}$	...	$\frac{(BH^2 - bh^2)^2 - 4BHbh(H - h)^2}{6(BH^2 + bh^2 - 2bHh)}$

LECTURE XXXII.—QUESTIONS.

1. A wrought-iron flanged girder is required to support a travelling load of 50 tons, the distance between the supports being 40 feet. The stress comes upon the girder at two points, the wheels on the traveller being 10 feet apart. What section of girder will be required to afford the necessary strength, presuming that the ultimate strength of the girder is six times that of the greatest stress to which it will be subjected?

2. Prove the law which governs the transverse strength of a beam of timber when supported at both ends and loaded at the centre. How are the constants required for applying this law arrived at?

3. A bar of wood, 7 feet long and 2 inches square, is supported at both ends, and is broken by a weight of 500 lbs. suspended at the centre. What weight in pounds will a rectangular bar of the same material, supported and loaded in like manner, sustain, when its length is 8 feet, its breadth $2\frac{1}{2}$ inches, and its depth 4 inches? *Ans.* 2187·5 lbs.

4. A rectangular beam of fir, of uniform section throughout, is supported horizontally on two walls 15 feet apart, and has to carry a load of $1\frac{1}{2}$ tons at 5 feet from one of the walls. The width of the beam is 5 inches; find its depth, taking the breaking load at four times the safe load. How much should the depth of the beam be increased, the breadth remaining constant, if the load were shifted from its original position to the centre of the beam, the breaking weight of a beam of fir 15 inches long, 1 inch broad, and 1 inch deep, supported at both ends and loaded in the middle, being taken at 360 lbs.? *Ans.* 8·9 inches; $\frac{1}{2}$ inch.

5. A solid rectangular girder, 3 inches deep and 2 inches broad, is supported at both ends on supports 5 feet apart. It is loaded with a uniformly distributed load, including its own weight, of 10 cwts. per foot run. What is the maximum intensity of stress at the outer fibres?

6. If two cast-iron beams—one circular in section and 2·73 inches in diameter, the other of rectangular section, 3 inches broad and 2 inches deep—be each supported at two points 20 inches apart, and loaded at the centre with a load of 2 tons; what will be the maximum intensity of stress produced in each case?

7. A beam of fir is built into a wall at one end, and projects 6 feet from the wall. The width of the beam is 4 inches; find its depth to bear safely a load of 1,200 lbs. uniformly distributed along its length. Assume that a bar of fir 1 foot long, 1 inch broad, and 1 inch deep, will break under a load of 125 lbs. when fixed at one end and loaded at the other end, and that the safe load is $\frac{1}{5}$ the breaking load. *Ans.* 6 inches.

8. What must be the breadth in inches of an oak cantilever or overhanging beam, 6 feet long and 9 inches deep, in order to carry a load of $\frac{1}{2}$ ton at its extremity, and how much must its breadth be increased in order that it may carry an additional load of $\frac{1}{2}$ ton uniformly distributed over its length? The actual stress is not to exceed $\frac{1}{4}$ of the breaking stress, and the breaking weight of an oak cantilever 6 inches long, 1 inch deep, and 1 inch broad, is 280 lbs. *Ans.* 2·37 inches; 1·18 inches.

9. A beam of fir supported at each end is inclined at an angle of 60° to the horizon, and is loaded at the centre of its length with a weight of 1 ton. The length of the beam is 10 feet, and its breadth is 2 inches, find the depth; the breaking load on the centre of a beam 1 foot long, 1 inch

broad, and 1 inch deep, and supported at the ends in a horizontal position, being 450 lbs. *Ans.* 3·527 inches.

10. A cast-iron cantilever or overhanging beam of T-section is 6 feet long, and 9 inches deep, the top flange being 6 inches wide. The beam has to carry, with safety, at its end a load of 1 ton, together with a distributed load of 1 ton over its length. Find the thickness of the top flange, the tensile breaking strength of cast iron being 8 tons per square inch, and the admissible load for a safe stress being one-fourth the breaking load. *Ans.* 1 inch.

11. Find the greatest load that may be uniformly distributed on a cast-iron girder having top and bottom flanges united by a web of the following dimensions—width of upper flange 3 inches, of lower flange 9 inches, total depth 12 inches, thickness of each flange and of the web being 1 inch, distance between the points of support 10 feet—when the greatest admissible stress in the compression flange is 3 tons per square inch, and that in the tension flange is $1\frac{1}{2}$ tons per square inch. *Ans.* 9·9 tons.

12. Make a diagram of a flanged cast-iron girder to carry a load of 12 tons in the centre, the distance between the points of support being 20 feet. What should you make the depth of the beam, and what should be the sectional area of the top and bottom flanges respectively?

13. A rolled steel girder has a mean depth of 10 inches, the top and bottom flanges are each 6 inches wide, and the metal in the flanges and webs is $\frac{1}{2}$ inch in thickness throughout. If the breaking strength of the material be taken as 40 tons to the square inch of section for both tension and compression, then (using 4 as a factor for safety) what would be the maximum safe load uniformly distributed over such a girder, supposing it to be supported at each end, the supports being 12 feet apart? Also make a diagram showing the distribution of the shearing stress in the middle transverse section. (S. & A. Hons. Exam., 1890.)

14. A rectangular beam of timber is supported at both ends, and loaded by a weight in the centre. Make the necessary calculation for measuring the strength of the beam to resist breaking. For a bar of larch 6 feet long by 2 inches square, supported as above, the breaking weight is 515 lbs.; taking this datum, you are required to solve the following question:—A cistern containing 2 tons of water rests on two cantilevers of larch, each 4 feet long and 5 inches in depth; find the breadth of each cantilever. *Ans.* 1·85 inches.

15. A cast-iron beam of rectangular section, and having its lowest side horizontal, is supported at both ends. What difference should you make in the upper outline according as the load is evenly distributed or collected in the centre?

16. A beam will safely carry a stationary load of 5 tons with a deflection of 2 inches, from what height may a weight of 200 lbs. be let drop upon the same beam without deflecting it to a greater extent? (S. & A. Exam., 1887.) *Ans.* 56 inches.

17. A steady load of 10 tons, suspended at the centre of a beam, deflects it through $\frac{5}{8}$ inch. From what height would a weight of 300 lbs. require to fall in order to produce a like deflection when dropping on the beam? (S. & A. Exam., 1891.) *Ans.* 22·7 inches.

18. A cylindrical iron beam is 15 inches in its external diameter, and the metal is $1\frac{1}{2}$ inches in thickness. The beam is fixed at the two ends, and is 35 feet between the supports; find the greatest load uniformly distributed that the beam will bear, the greatest safe stress on the metal being 9,000 lbs. per square inch. (S. & A. Hons. Exam., 1894.)

19. Compare the resistance to bending of a wrought-iron I section beam

when the beam is placed like this I, and like this ⊢⊣. The flanges of the beam are each 6 inches wide and 1 inch thick, and the web is $\frac{3}{4}$ inch thick and measures 8 inches between the flanges. (Adv. S. & A. Exam., 1897.)

20. A horizontal bar of round iron, 1 inch diameter, 6 feet long, hinged at the ends, is subjected to equal and opposite pushing forces of 1,000 lbs. at its ends, and a load of 10 lbs. is hung at the middle so that it is both a beam and a strut. Find the greatest stress anywhere. $E = 29 \times 10^6$ lbs. per square inch. (S. & A. Hons. Exam., 1897.)

21. Draw the bending moment diagrams, and state the maximum bending moments for the six standard cases of loading and supporting a beam of the same length, same load. (1) Fixed at one end, loaded at the other. (2) Fixed at one end, loaded uniformly. (3) Supported at the ends, loaded in the middle. (4) Supported at the ends, loaded uniformly. (5) Fixed at the ends, loaded in the middle. (6) Fixed at the ends, loaded uniformly. (Adv. S. & A. Exam., 1897.)

22. A uniform beam is fixed at its ends, which are 20 ft. apart. A load of 5 tons in the middle; loads of 2 tons each at 5 ft. from the ends. Find the diagram of bending moment and prove your rule. State what the maximum bending moment is, and where are the points of inflexion. (Hons. S. & A. Exam., 1897.)

The following is a list of books on the Strength of Materials:—

Experimental Engineering, by Rolla C. Carpenter, C.E. (Chapman & Hall, London, 1895.)

Theory of Structures and Strength of Materials, by Henry T. Bovey, M.A., D.C.L. (John Wiley & Sons, New York.)

The Practical Strength of Beams. Paper by B. Baker. (Proc. Inst. C.E., vol. lxii., p. 251.)

Mechanical Engineering Materials, by E. C. R. Marks. (Technical Publishing Co., Manchester.)

Strength and Properties of Materials, by W. G. Kirkcaldy, London.

Strength and Determination of the Dimensions of Structures of Iron and Steel, by J. J. Weyrauch, translated by A. J. Du Bois. (New York, 1891.)

Strength of Materials and Structures, by Sir J. Anderson. (Longmans, Green & Co., 1892.)

PART VI.—HYDRAULICS AND HYDRAULIC MACHINERY.

LECTURE XXXIII.

CONTENTS.—Labour-Saving Appliances in Modern Gas Works—Pumping Engines and Accumulator—Example I.—Differential Accumulator—Brown's Steam Accumulator—Small Hydraulic Accumulator Plant—Arrol-Foulis' Gas Retort Charging Machine—Foulis' Withdrawing Machine for Gas Retorts—Results of Working—Questions.

Labour-Saving Appliances in Modern Gas Works.*—In no industry have the conditions of labour undergone more radical and rapid changes than in large modern gasworks. Until recently, the coal was emptied by hand from railway waggons or from carts at the most convenient position for the retort benches. If the lumps of coal were too large, they were broken up by manual labour, and the retorts were charged, as well as discharged, by hand. The surplus coke, beyond what was required for the furnaces, was quenched, wheeled into a yard, and there screened and filled into carts or waggons, all by hand labour.

In modernised works, the coal is usually delivered direct from the railway truck into the hopper of the mechanically-driven coal breaker. After passing through this machine, it is raised by an elevator to a large hopper, which is so fixed that the coal can be automatically delivered into the hopper of any of the charging machines. The charging of the retorts is then performed by means of a hydraulic charging machine more evenly and otherwise better than by hand. After carbonisation, the coke is withdrawn from the retorts by a hydraulic drawing machine. These two operations now entail a minimum of labour and inconvenience from heat to the attendants. The surplus coke, guided in its descent by shoots, falls into a bogie underneath the floor, and is run out to the yard by small locomotives. In some instances conveyors, which are placed underneath the stage floor, carry the coke to circular revolving screens, whence it is delivered into large storage hoppers, railway waggons, or bags for small consumers.

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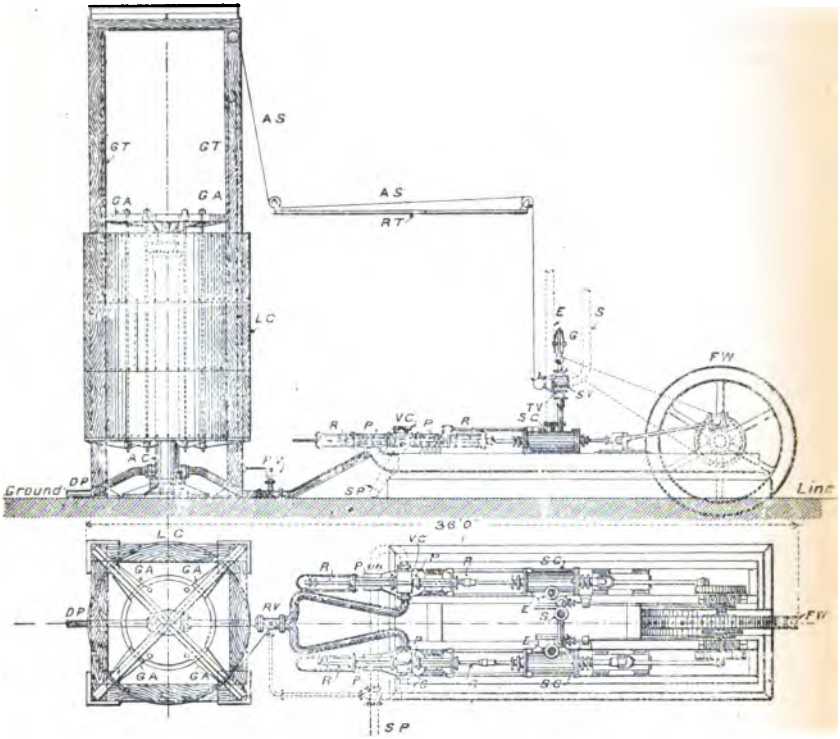
* I am indebted to Mr. Andrew S. Biggart, of Sir William Arrol & Co., Ltd., Glasgow, for the drawings from which the three electros of the General Arrangement, Charging, and Drawing Machines were made. I am also indebted for the information contained in Mr. Biggart's paper which he read before the 1895 Glasgow meeting of the Institution of Mechanical Engineers, as well as to Mr. Foulis, the General Manager of the Glasgow Corporation Gas Trust, for showing me the whole of the plant in operation.

The old methods involved continuous and repeated operations performed by hand; while the new are such that no hand labour is employed in dealing with the coal from the time it leaves the railway truck until it arrives there again in the form of coke. Hand labour is thus entirely superseded by mechanical power, to the great advantage of both the labourer and the employer. The number of men required in the retort house under the new system is less than half that required under the old method. The saving which this represents, after allowing for maintenance of plant and interest on additional capital, will average about one shilling per ton.

Pumping Engines and Accumulator.—The general arrangement of this portion of the plant is illustrated by the following side elevation and plan. The steam engine and pumps are situated either in the main engine room of the gasworks or in a convenient house not far removed from the steam boilers; while the accumulator may be placed in an annex, where there is plenty of head room for the guide timbers. From what was said about the construction and action of the Armstrong accumulator in Lecture XX. of author's *Elementary Manual on Applied Mechanics*, and by carefully comparing the present figures with the index to parts (which has been tabulated in nearly the exact sequence of the operations) the student will have no difficulty in understanding the generating plant which serves to supply hydraulic power to the charging and drawing machines in the retort houses.

The steam engine drives the rams R of the pumps P direct from a back extension of the piston-rods. From the pumps the water is forced into the accumulator cylinder A C at such a pressure that, acting on the accumulator ram, it is capable of lifting the heavy dead-weight bolted to the upper end. This weight is provided with guide arms G A, bearing upon planed iron rails fixed to the vertical guide timbers G T. When the guide arms have reached a certain height, the one at the right hand begins to lift a weight attached to a chain on the weighted lever of the throttle valve T V. During the remainder of the ascent of the accumulator the automatic starting chain A S is slackened until the throttle valve is closed, and the engine is stopped before the guide arms reach the cross beam, which connects the upper ends of the guide timbers. Should the chain stick, or anything fail about this automatic system of stopping the engine, then a second chain, which connects the weight at the end of the relief valve R V to the counterpoise ball



General Arrangement of Pumping Engines and Accumulator for Working Gas Retort Charging and Withdrawing Machines.

Index to Parts.

S	for	Steam Pipe.	V C	for	Valve Chest for Pumps.
S V	,,	Stop Valve.	R	,,	Rams.
T V	,,	Throttle Valve.	P	,,	Pumps.
A S	,,	Automatic Starting and Stopping Apparatus.	R V	,,	Relief Valve.
S C	,,	Steam Cylinder.	A C	,,	Accumulator Cylinder.
E	,,	Exhaust Pipe.	D P	,,	Delivery Pipe.
G	,,	Governor.	G A	,,	Guide Arms.
F W	,,	Flywheel.	G T	,,	Guide Timbers.
S P	,,	Suction Pipe.	L C	,,	Load Casing.
			R T	,,	Roof Tie-Rods.

attached to the chain A S, is tightened up just before the guide arms reach the uppermost limit of their travel.* This action releases the downward pressure on the relief valve, and permits sufficient water to escape to prevent any further elevation of the accumulator ram. The governor G now prevents the engine from racing. When water is used by any of the hydraulic machines it flows to them from the accumulator cylinder through the delivery pipe D P. This allows the accumulator ram and weight to sink until the ball attached to A S re-asserts its pull on the throttle valve, and, automatically opening it, starts the engine.

It is evident that if a set of engines and pumps were devoted to the direct supply of high-pressure water to several machines, their output would have to be equal to the greatest requirements of the plant at any instant; but by the introduction of the accumulator and its automatic starting and stopping gear we have a simple means of ensuring a constant supply of water at the desired pressure with a smaller engine.

An accumulator, therefore, performs several very important functions in a most efficient manner:—

(1) It acts as a reservoir for the storage of energy.

(2) It acts as a regulator of pressure.

(3) It acts like a flywheel in taking up and giving out power in direct sympathy with the immediate wants of supply and demand.

(4) It acts as an elastic buffer, and prevents the breakage of joints, &c.

(5) It automatically controls the motive power.

The efficiency of an accumulator has been proved to be as high as 98 per cent., only 1 per cent. being lost through friction in charging, and 1 per cent. in discharging it, as tested by pressure gauges on the supply and discharge pipes. Its total store of energy is, however, comparatively small, since it is only equal to the potential energy of the weight raised. Hence, if W lbs. be the total weight raised in the accumulator, and H feet the difference of height between its highest and lowest positions, we have:—

$$\left.\begin{array}{l}\textbf{Energy Stored in}\\ \textbf{Accumulator}\end{array}\right\} = \mathbf{W\,H\ ft.\text{-}lbs.} = \frac{\mathbf{W\,H}}{\mathbf{33{,}000 \times 60}}\ \mathbf{H.P.\text{-}hours.}^{\dagger} \qquad \text{(I)}$$

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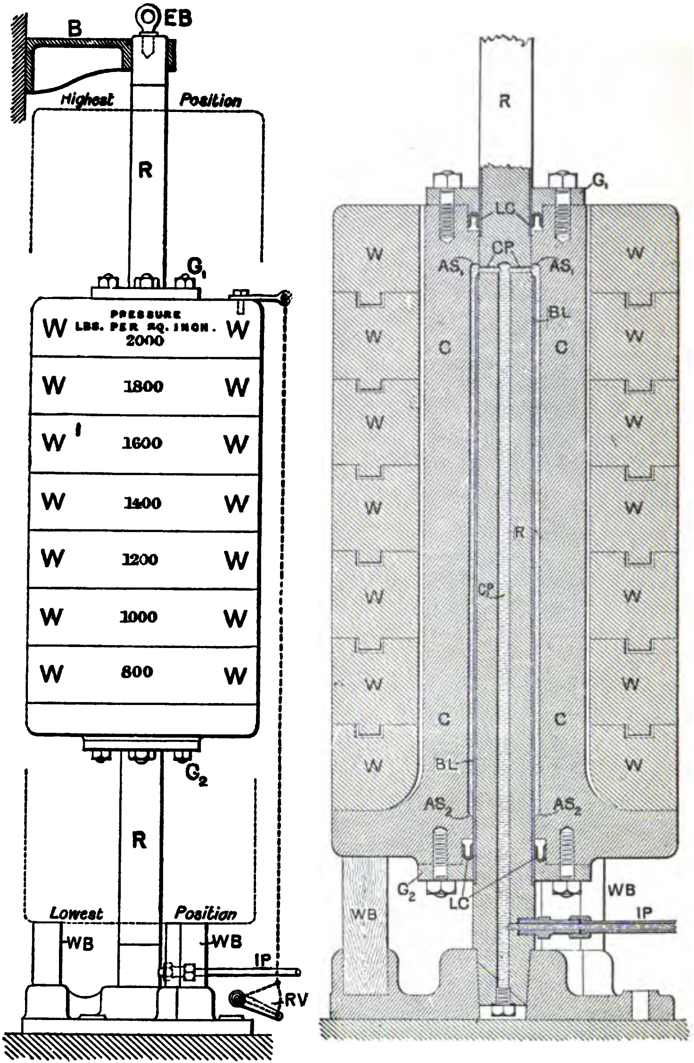
* Another arrangement is to pass this second vertical chain through a hole in a projecting plate fixed to the right-hand guide arm G A, and then attach its upper end to the top cross beam. If the accumulator should rise too high, then the plate catches an enlarged portion of the chain and opens the relief valves.

† One horse-power = 33,000 ft.-lbs. of work per minute, hence, 1 horse-power hour is 33,000 × 60 or 1,980,000 ft.-lbs.



ELEVATION.

VERTICAL SECTION.

TWEDDELL'S DIFFERENTIAL ACCUMULATOR.

Example I.—The accumulators used in connection with the hydraulic power supply in Glasgow are 18 inches in diameter, and have a free lift of 23 feet. The total load on each is 127 tons. Find the pressure of the water and the maximum energy stored in each accumulator, neglecting friction.

Answer.—

Pressure of water $= p = \frac{W}{A}.$

,, ,, $= \frac{127 \times 2{,}240}{\cdot 7854 \times 18 \times 18} =$ **1,120 lbs. per sq. in.**

Energy stored $= W H = 127 \times 2{,}240 \times 23.$

,, $= 6{,}543{,}000$ ft.-lbs., or **3·3 H.P.-hours.**

Index to Parts.

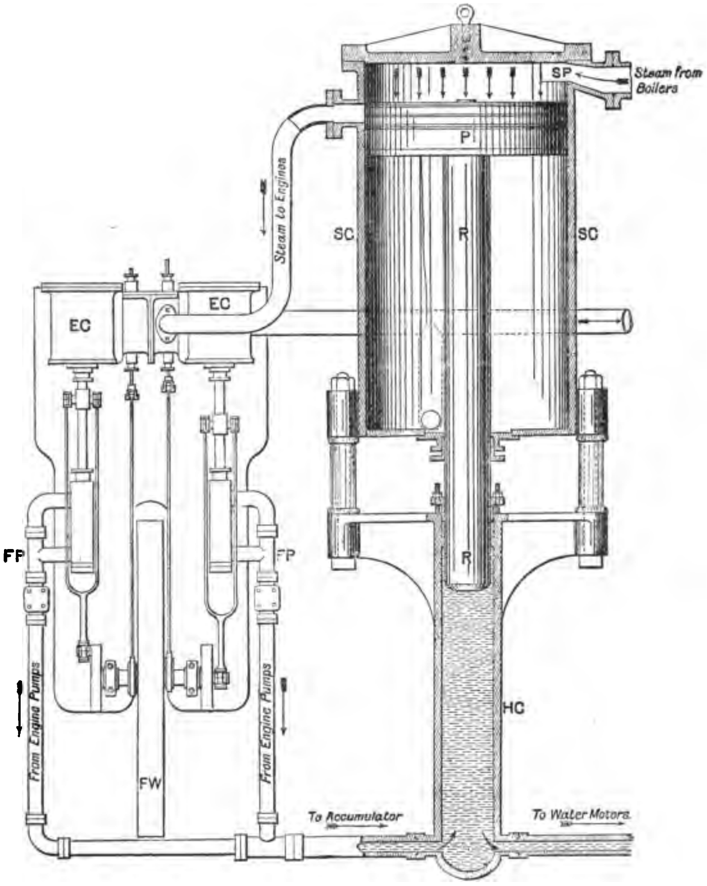
I P	for	Inlet Pipe from pump.
R	,,	Ram of accumulator.
B L	,,	Brass Liner on lower part of R.
C P	,,	Central and Cross Passages for water.
C	,,	Cylinder.
AS_1, AS_2	,,	Annular Spaces from top and bottom of cylinder.
G_1, G_2	,,	Glands (top and bottom).
L C	,,	Leather Cup Packings.
W	,,	Weights.
W B	,,	Wooden Blocks.
B	,,	Bracket (at top).
E B	,,	End Bearing.

It will be seen from this example how small the total store of energy is, and that accumulators would be quite useless to maintain the supply for any length of time. One of the real advantages of the accumulator arises from the fact that we may use its energy for a very short time at a high rate. For instance, although the accumulator in the above example is only capable of maintaining 3·3 H.P. for a whole hour, it could exert 19·8 H.P. for ten minutes, or 198 H.P. for one minute.

Differential Accumulator.—Although it has only been found necessary to employ a water pressure of 400 lbs. per square inch in the charging and drawing machines for gas retorts, which we will describe later on, and, further, since it is advisable with them to have a considerable volume of water in the accumulators when many machines have to be worked simultaneously, yet it may not be out of place to describe here the differential accumulator designed by Mr. Tweddell for his smaller kinds of hydraulic tools, since there may be cases in which space and compactness are of considerable importance. From the foregoing elevation and enlarged vertical section it will be seen that the ram R consists of a vertical fixed shaft secured at the top by a bracket B, and at the bottom by a footstep. The lower half of this shaft is of larger diameter than its upper half, a brass liner B L being shrunk on the former part. Moreover, this lower portion of the ram has a central passage C P drilled axially along it, with a cross passage just above the upper end of the brass liner. Through these passages water is admitted from the inlet pipe I P, which is connected directly to the force pumps. This water finds its way into an annular space A S_1, A S_2, which is the clearance between the outside of the brass liner and the inner bore of the heavy press or cylinder C. Surrounding the outside of the cylinder are placed a number of cast-iron or lead weights W which fit into each other, and form the dead load along with the weight of the cylinder. At the top and the bottom of the cylinder there are suitable glands G_1 and G_2 containing the usual leather cup packings L C. When the machine is idle the bottom flange of the cylinder rests upon wooden beams W B. It will now be readily understood that the effective area of the ram is *only* the difference between the cross areas of the brass liner B L and the upper part of the ram R, instead of the whole area of the ram as in the previous case. Hence, a very great pressure may be obtained from a small weight. For example, should the annular area representing the difference in size between the brass liner and the upper part of the iron ram be 5 square inches, and the total weight of the cylinder and its surrounding cast-iron blocks be 2,000 lbs., then, neglecting the friction at the glands, the pressure would be 2,000 ÷ 5, or 400 lbs. per square inch. This accumulator will store up an equal amount of energy, as in the previous case, if the dead weight and height of the lift are the same since their stored energy depends directly upon W × H. The volume of water contained in the accumulator will, however, be comparatively small, and hence it will fall more quickly for a certain amount of water used by the hydraulic machines which it drives. As will be seen from the left-hand figure, a relief valve R V is worked by a chain connecting an outstanding arm on the uppermost weight to the end of the lever; also any desired pressure up to 2,000 lbs. per square inch, or more, may easily be obtained from this accumulator with a comparatively small dead load and space.

Brown's Steam Accumulator.—Another very simple form of accumulator, which has proved very effective both for land purposes and on board ship, is that designed and made by Mr. A.

Betts Brown, of the Rosebank Iron Works, Edinburgh. It consists of a steam cylinder S C fitted with a piston P and a piston-rod or ram R. Steam is supplied direct to this cylinder from the boiler and presses on the piston P in opposition to the



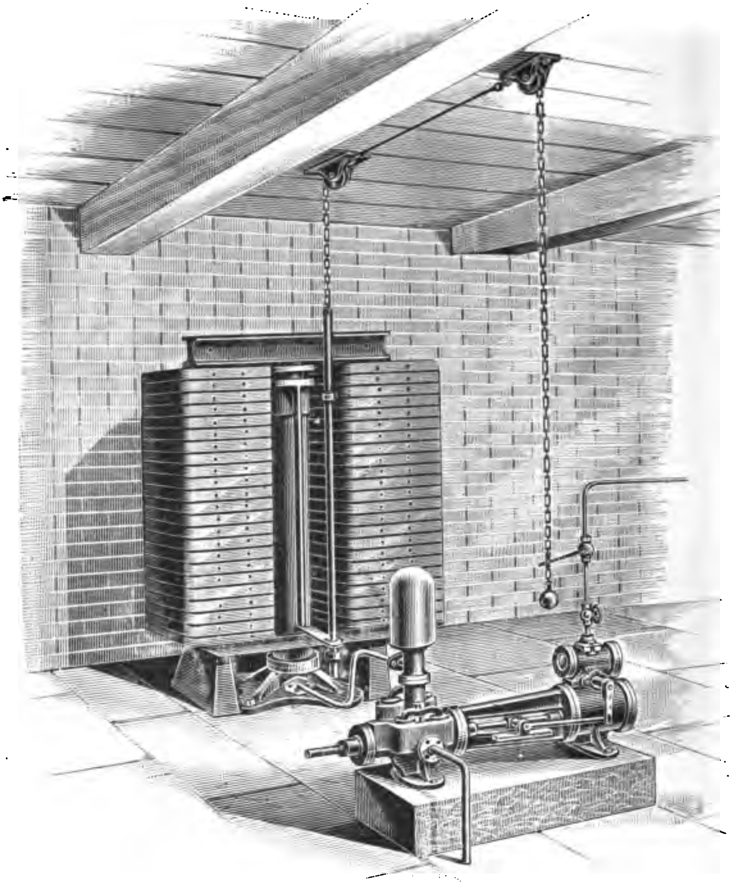
BROWN'S STEAM ACCUMULATOR.

INDEX TO PARTS.

S P for Steam Pipe from Boilers.
S C ,, Steam Cylinder.
P ,, Piston working in S C.
R ,, Ram attached to P

H C for Hydraulic Cylinder.
E C ,, Engine Cylinders.
F P ,, Force Pumps.
E ,, Exhaust Pipe.

water forced into the hydraulic cylinder H C by the force pumps F P, which are worked by a pair of engines. An exhaust pipe E carries away the exhaust steam from the engine cylinders E C and the bottom of the large steam accumulator cylinder S C.

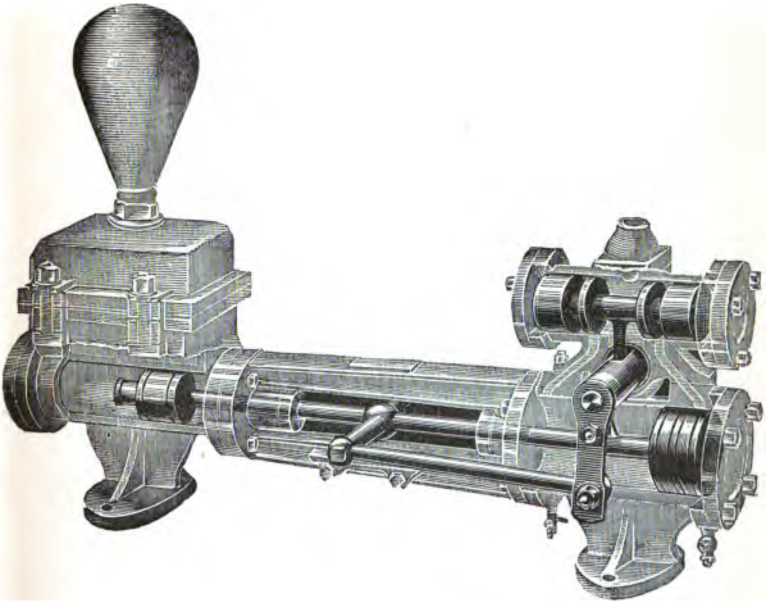


SMALL HYDRAULIC ACCUMULATOR PLANT.

Suppose that the piston P is at the bottom of its cylinder, then the boiler steam not only fills the portion above the piston but passes on to the engine cylinders and therefore works the pumps.

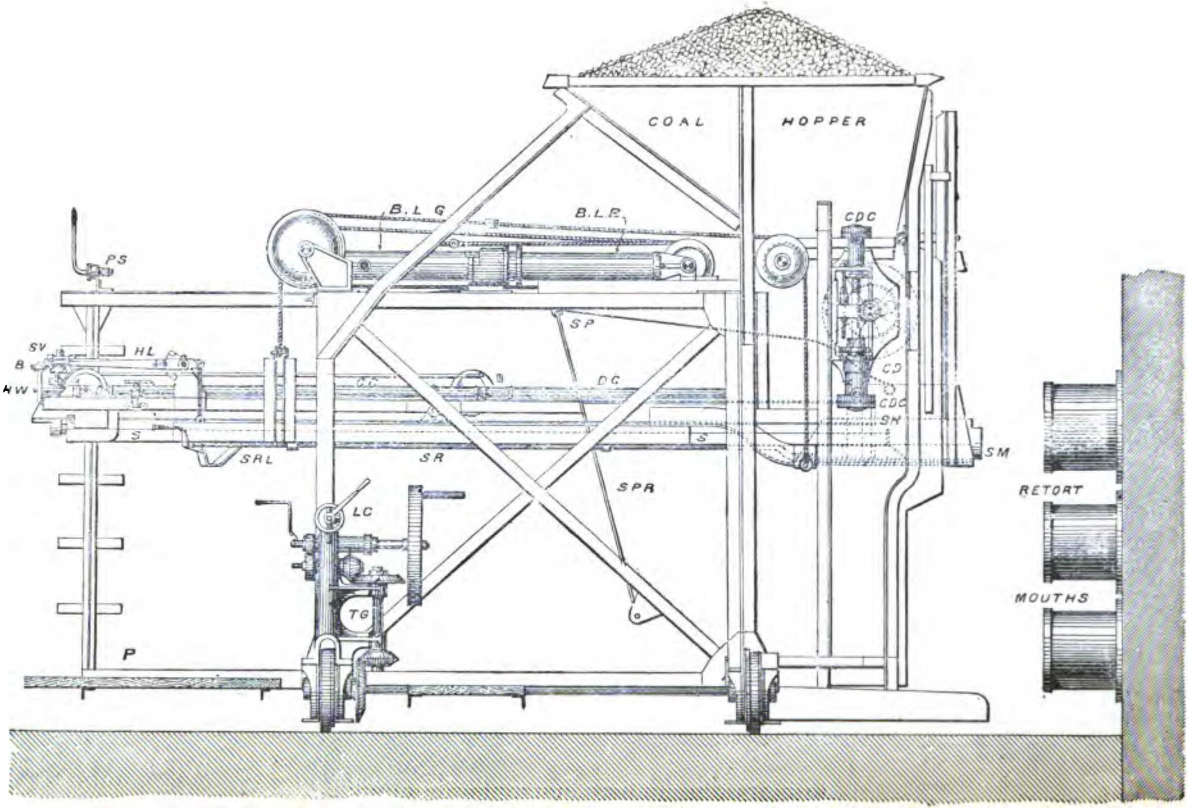
and forces up the ram and piston to the top of their stroke, when the piston gradually closes the steam passage leading to the engines until they stop. Should any of the hydraulic machines be now put into action the water flows to them from the hydraulic cylinder, the ram and piston descend and the engines are again set into motion to keep up the demand and again close the engine steam pipe. With a steam cylinder of 36 inches diameter and a ram of 9½ inches (or, ratio of areas 15 to 1) Mr. Brown is able with about 50 lbs. steam pressure per square inch to maintain a pressure of 750 lbs. per square inch in the hydraulic mains leading to water motors, steering, stopping, and starting gear, or, in the case of a gas works, to the charging and drawing machines.

Small Hydraulic Accumulator Plant.—Should the gasworks be a small one, then a much less expensive accumulator plant may be employed, such as that illustrated. This figure is sufficiently self-descriptive after what has been said about the larger plant.



Direct Acting Steam Pump.

The small pumping engine, or donkey pump, which supplies water of the desired pressure to the accumulator is also shown in a perspective view. Steam is admitted to the steam



Arrol-Foulis' Gas Retort Charging Machine.

cylinder by piston valves, actuated by a connecting-rod and lever, while the double-acting pump is fitted with a solid piston plunger and an air vessel.

Arrol-Foulis Gas Retort Charging Machine.—In replacing hand labour for the charging gas retorts by a hydraulic machine several important requirements have to be effected:—

(1) The machine should be easily moved parallel to the retort bench at the proper distance therefrom.

(2) It must be capable of being connected with the hydraulic pressure pipe at every position.

(3) The shoot mouth should be easily raised or lowered to suit the highest or lowest retort.

(4) The shoot mouth should be easily entered into the mouth of the retort before filling it with coal.

(5) The coal hopper should contain at least a sufficient quantity of coal to charge a complete set of retorts without requiring to be refilled.

INDEX TO PARTS.

P	for	Platform (for attendant).	C D	for	Coal Drum.
T G	,,	Travelling Gear.	S V	,,	Slide Valve.
P S	,,	Pressure Swivel.	C C	,,	Charging Cylinder.
L C	,,	Lifting Cock.	S	,,	Spear.
B L C	,,	Beam Lifting Cylinder.	S H	,,	Spear Head.
B L R	,,	Beam Lifting Ram.	D C	,,	Drawing Cylinder.
S R L	,,	Shoot Rod Lever.	S P	,,	Swing Plate.
S R	,,	Shoot Rod.	S P R	,,	Swing Plate Rods.
S M	,,	Shoot Mouth.	H W	,,	Hand Wheel.
H L	,,	Hand Lever.	B	,,	Bell.
C D C	,,	Coal Drum Cylinders.			

(6) The exact quantity of coal required for the charge of each retort should be easily regulated, and the regulator should be capable of dealing with all the sizes of coal in use.

(7) The charge should be laid evenly and of the desired depth from back to front without any special effort on the part of the attendant, and without loss of time.

(8) The platform of the attendant should be placed in the most convenient position for actuating all the motions, and at the same time protecting him from the heat emanating from the retorts.

(9) The whole machine should require a minimum of attention and repair. It should also be able to withstand the rough usage of retort house workmen, as well as the hydraulic pressure, without undue leakage or giving way in any of its vital parts.

These several requirements have been very ingeniously worked out and applied by Sir William Arrol and Mr. William Foulis in their patent hydraulic charging machine, shown by the accompanying figure, as follows:—

(1) The whole machine with its load of coal is supported on a pair of rails, placed parallel to the retort bench, and moved by travelling gear T G. In the illustration ordinary hand gear is shown, but in the latest machines a hydraulic motor of special design has been applied with excellent results.

(2) A pipe is fixed the whole length of the retort bench at a convenient height and distance therefrom, and it is always kept charged with water from the accumulator at the full working pressure of 400 lbs. to the square inch. At suitable distances along this pipe there are attached ordinary armoured flexible hose pipes, which may be connected to the pressure swivel P S. From this swivel copper pipes lead to the lifting cock L C and slide valve S V.

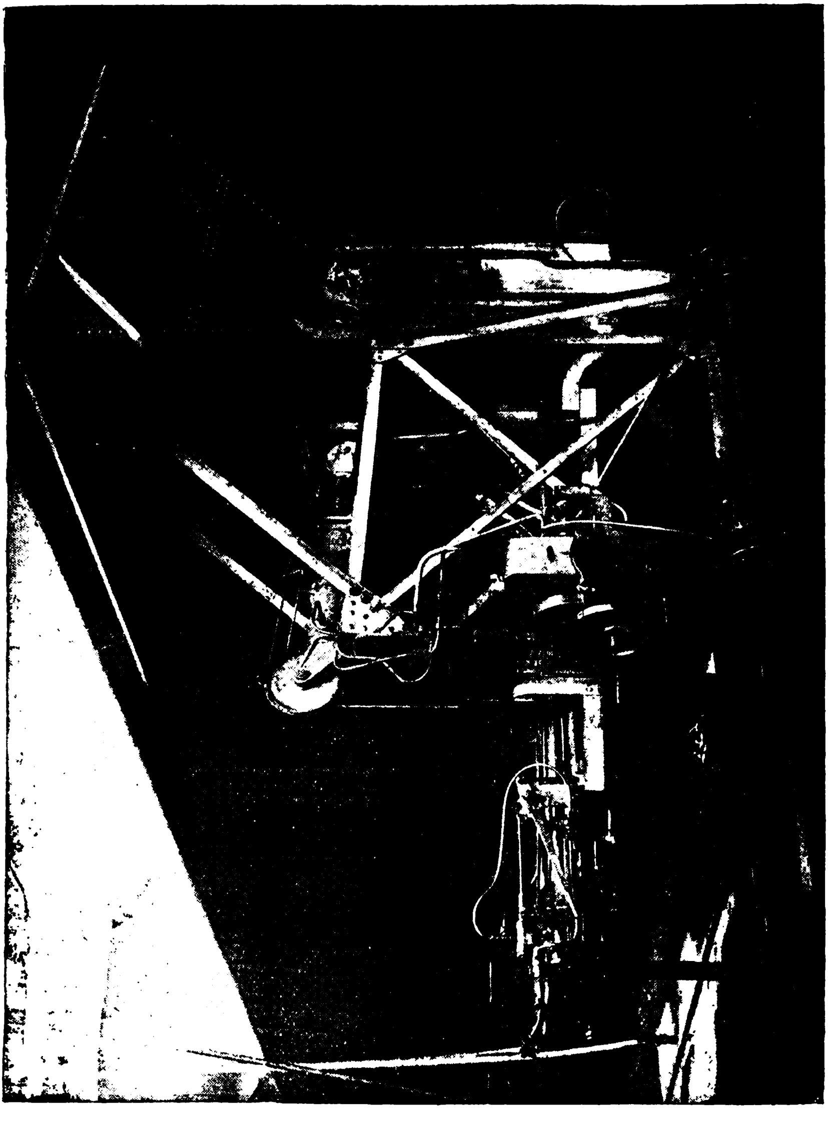
(3) The beam carrying the shoot mouth, shoot rod S R, spear S, charging cylinder C C, and drawing cylinder D C, is raised and lowered by turning the lifting cock L C to the right or left and thereby admitting water to the beam lifting cylinder B L C.

(4) When the shoot mouth S M is fairly opposite one of the retort mouths it is pushed forward by the shoot-rod lever S R L, and the hand lever H L is turned to admit water to the slide valve S V and work the charging cylinder C C, so as to push the spear S and spear head S H into the retort, having a quantity of coal in front.

(5) The coal hoppers are made to contain from 2 to 5 tons.

(6) The feeding gear consists of an open coal drum C D, which is divided into segmental compartments, each of which can contain a certain quantity of coal. It is turned through one or two divisions at regular intervals by a hydraulic ram with rack and ratchet gear, so as to permit the desired quantity of coal to fall into the shoot. A plate, which acts as a flap valve, is so fixed in front of the drum by a lever and weight, that it presses against the face of the drum, and prevents the small coal from falling down past the face of the drum.

(7) The coal falls from the hopper in front of the pusher plate or spear head S H, and is delivered into the retort by a series of six or seven successive strokes, each stroke being shorter than the previous one. This is accomplished by means of a shaft carrying a set of stops placed on the beam alongside the spear S. This shaft is automatically turned a certain amount during each return stroke, so as to bring the stops into position in rotation. The forward and return strokes of the spear are caused by two



General View of Arrol-Foulis Charging Machine.

hydraulic rams working in the cylinders C C and D C, which are regulated by the hand lever H L. This lever also serves to lower the spear head when entering the retort for pushing in the coal, and to raise it clear from the bottom of the retort during the return stroke. It also causes the revolution of the stopper shaft. The inclined form of the spear head and these successive strokes ensure the charge being evenly distributed along the retort, and the depth is regulated by putting in more or less coal at each stroke, as previously explained.

(8) The platform P is placed at the back of the machine, so that the attendant has the levers within easy reach, and he is placed as far as possible from the heat of the retorts.

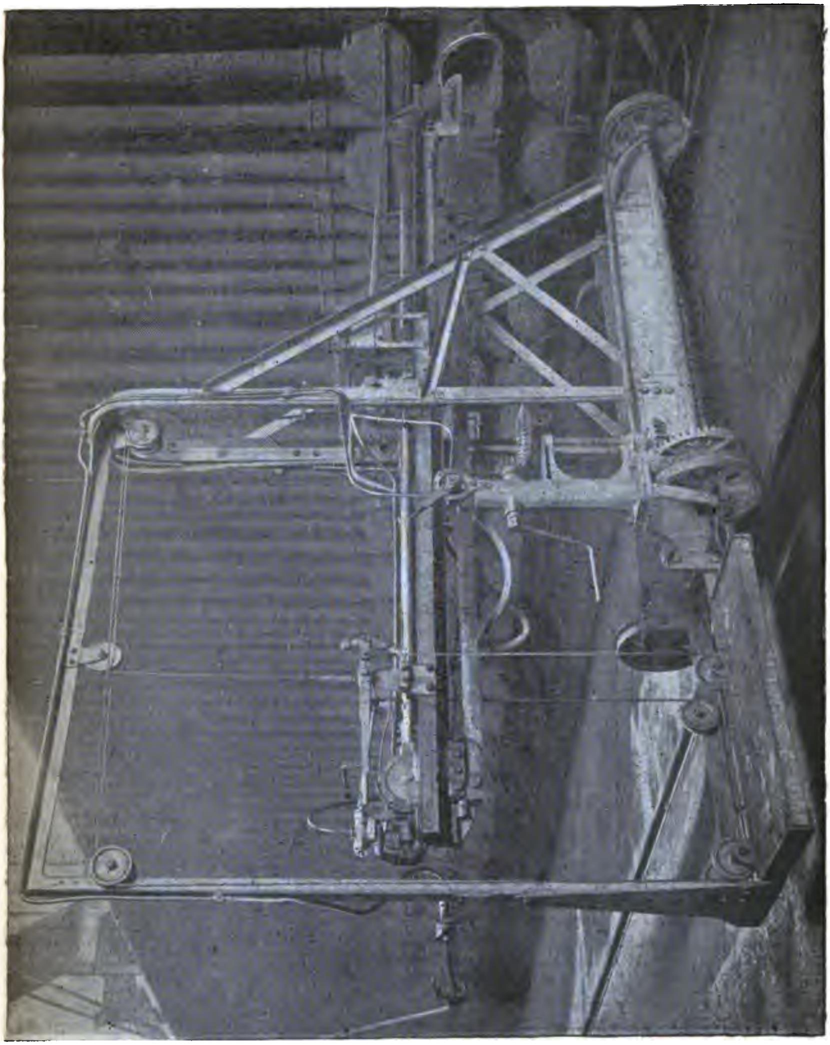
(9) Great attention has been given to the several details of the machine, so as to render it as durable as possible. The reason for adopting this comparatively low pressure is, that the power required for the different motions is so small that if a higher pressure were employed the rams would be inconveniently small. In fact, with 400 lbs. pressure it had been found in some instances that rams of only $1\frac{1}{4}$ inch diameter gave ample power.

Foulis' Withdrawing Machine for Gas Retorts.—After the coal has been carbonised and converted into coke, it has to be withdrawn from the retorts before putting in a new charge. This is done by a machine which is similar to that used for charging, and travels along the same rails. The withdrawing machine consists of a frame supported on a truck and carrying a rake for pulling out the coke. The head of this rake can swing backwards into a horizontal position so as to clear the coke when being moved inwards, but goes back to the vertical and dips into the coke on its return stroke.

The beam B carrying the rake can be raised or lowered, and the rake moved out and in, in the same manner as the corresponding parts of the charging machine. As, however, the withdrawing machine is comparatively light, it is found quite sufficient to use hand travelling gear T G to make it move along the rails.

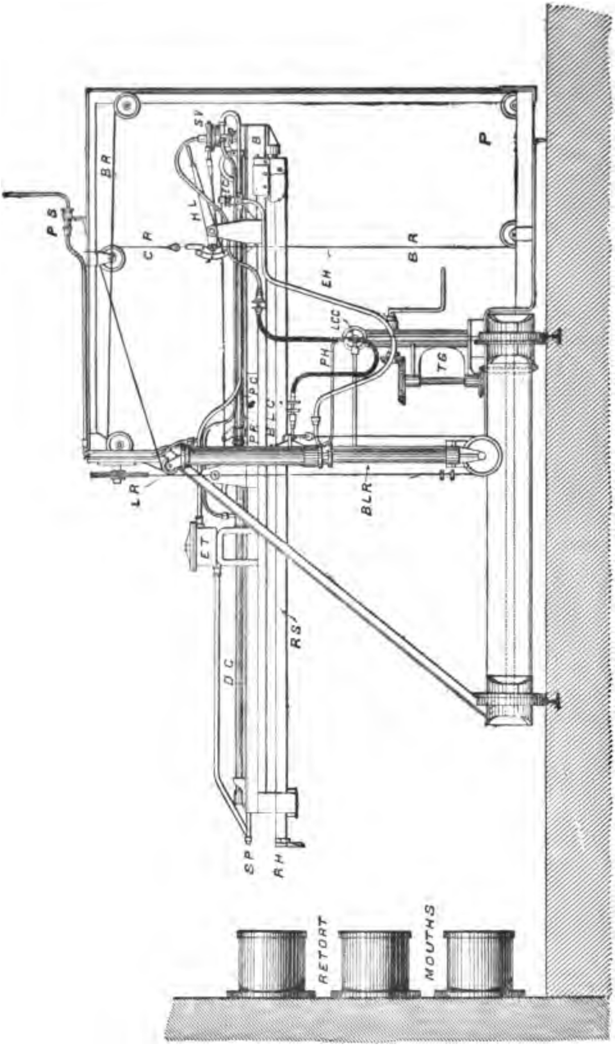
The exhaust water is led to a tank E T, from which it flows to a spray pipe S P and plays on the rake head to keep it cool and help to quench the coke.

Results of Working.—Mr. Biggart says that the number of retorts charged or drawn per hour by these machines varies to a considerable extent in actual work. In some cases, owing to special circumstances, not more than twenty-four per hour are available for each machine; while in other more favourable instances as many as forty-eight per hour are allotted to each, and even with this larger number a reasonable time remains for



Foulis' Withdrawing Machine for Gas Retorts.

rest for the attendants at the end of each hour. The labour of charging the retorts and withdrawing the coke is much lightened by these mechanical means, and the number of retorts charged and drawn for each man employed is largely increased. It



Foulis' Withdrawing Machine for Gas Retorts.

might at first be imagined that coal placed in retorts in only six to eight large charges by the machine would not be so evenly laid as a much larger number of smaller charges put in by hand. The machine, however, lays the coal by far the most evenly, owing partly to the shape of the pusher head, which is bevelled so as to allow the small ridge of coal raised in pushing forwards to fall back when its support is removed on the withdrawal of the pusher head. Another advantage possessed by machine work over hand labour is that the charging is done more quickly, and thus there is a diminished loss of gas before the retort doors are closed.

Apart from any other consideration, the mechanical charger could not fail to prove beneficial in view of the greatly improved conditons under which it enables work of a most trying nature to be carried on. The old method of hand charging was a severe ordeal for the stokers, requiring great exertion to get through the work in the shortest time possible, while exposed throughout to a high heat. Such adverse conditions are now entirely done away with where mechanical stoking obtains. The single lever by which the whole of the operations are controlled is worked from such a position that the attendant is quite removed from the discomfort of close proximity to a high heat, while at the same time the former severe bodily exertion is replaced by light and easy work. Even greater improvements in the conditions of labour arise from the introduction of the drawer, which accomplishes, under all the better conditions attending the use of the charger, work of a still more trying nature. The withdrawing of the hot coke from the retorts was work for which even the stokers themselves, accustomed as they were to it, admitted that mechanical appliances were required. Here again all is worked by a single lever, in such a position as to remove the attendant from the former discomforts of withdrawing the coke at a white heat at the mouth of the open retort.

Index to Parts.

T G	for	Travelling Gear.	P H	for	Pressure Hose.
P S	,,	Pressure Swivel.	P C	,,	Pushing Cylinder.
L C C	,,	Lifting Cylinder Cock.	P R	,,	Pushing Ram.
B	,,	Beam.	R S	,,	Rake Spear.
B L C	,,	,, Lifting Cylinder.	R H	,,	,, Head.
B L R	,,	,, ,, Ram.	E C	,,	Exhaust Cock.
L R	,,	Lifting Rope.	E H	,,	,, Hose.
B R	,,	Balancing Ropes.	E T	,,	,, Tank.
C R	,,	Check Rope.	S P	,,	,, Spray Pipes.
H L	,,	Hand Lever.	D C	,,	Drawing Cylinder.
S V	,,	Slide Valve.	P	,,	Platform (for Attendant).

LECTURE XXXIII.—QUESTIONS

1. Sketch and describe the general arrangement of pumping engines and accumulator as used for supplying hydraulic power to the mechanical stoking appliances in a gas works.

2. Make a vertical section of an accumulator, and explain the manner in which this apparatus enables us to store up and give out energy. If the ram of the accumulator be 17 inches in diameter, what should be the load in order to obtain a water pressure of 700 lbs. on the square inch? *Ans.* 70·9 tons.

3. Sketch and describe some arrangement of an hydraulic accumulator by which a pressure of 10 tons to the square inch can be obtained for testing purposes, with pumps working at a pressure not exceeding 3 tons to the square inch. (S. and A. Exam., 1890.)

4. Describe, without going into detail, the engines, pumps, accumulator, and one or two of the appliances likely to be used by customers of an hydraulic company. (S. and A. Adv. Exam., 1897.)

5. If you desire to obtain great pressure with a small dead load, what form of accumulator would you employ? Give a vertical section and a description of the construction and action of the accumulator you have selected. Why is it inadvisable to use one with a very small ram?

6. If you desire to obtain a pressure of 1,200 lbs. per square inch, and you are limited to a dead weight of 1 ton, what effective area would you require, and what would be the diameter of the larger part of the ram if the smaller be 6 inches in diameter, in the case of a differential accumulator having a total efficiency of 90 per cent.? What would be the diameter of the ram if you used a solid one?

7. Sketch and describe Brown's Steam Accumulator. Mention any advantages and disadvantages which you consider it possesses with respect to other forms of accumulator.

8. Suppose the effective steam pressure in the steam cylinder of a Brown's Accumulator to be 60 lbs. on the square inch, and that you require a water pressure of 1,000 lbs. per square inch, with a ram of 20 square inches in cross section, what will be the diameters of the ram and steam piston if 2 per cent. be lost in friction?

9. Sketch the construction and describe the action of the Arrol-Foulis hydraulic apparatus for charging gas retorts. Mention the several advantages of employing this machine as against hand labour.

10. Sketch the construction and describe the action of the Foulis hydraulic apparatus for withdrawing the coke from gas retorts. Mention the several advantages of employing this machine as against hand labour.

LECTURE XXXIV.

CONTENTS.—Refrigeration—Preliminary Considerations—Carbon Dioxide as a Refrigerating Agent—Elementary Refrigerating Apparatus—Simple Refrigerating Machine—Carbon Dioxide Refrigerating Plant—Anhydrous Ammonia as a Refrigerating Agent—De La Vergne's Refrigerating Plant—De La Vergne's Double Acting Compressor—The Linde System of Refrigeration—Apparatus for Transmitting the cold produced to the Chambers requiring Refrigeration—Questions.

Refrigeration—Preliminary Considerations.—An interesting example of the conversion of heat into work is afforded by a refrigerating machine. The simplest form of machine consists of an air-compression pump driven by a steam engine, or other motive power, in which the pump is water jacketted and the air is cooled under pressure by being passed through a surface condenser where the water abstracts the sensible heat generated by the mechanical work of compression. The air thus cooled, but still under pressure, is conveyed to an air engine and allowed to perform work by expanding against some resistance. A large proportion of the work originally performed during the operation of compression is again given out, with a corresponding reduction in the air temperature. A machine on this principle may be conveniently constructed by arranging the steam, compression and expanding engines to work on one crank-shaft. The expanding air thus assists in the work of compression. After deducting the necessary losses due to cooling, leakage, &c., the work done in the expansion cylinder amounts to about 65 per cent. of the power absorbed by the compression cylinder; the remaining 35 per cent. being supplied by the steam or other prime mover. The air having thus given up its heat, exhausts from the expansion cylinder at a very low temperature, reaching in one authenticated instance as low as -124° F.

The large coal consumption of machines of this class has, however, led to their being superseded, in almost all cases, by machines in which a more direct conversion of heat into work takes place.

If we take any liquid and commence to vaporise it, we find that, it is necessary to maintain a continual application of heat in order to bring about this physical change. The amount of heat necessary to convert a unit weight of a liquid to a unit weight of gas at the same pressure, is always constant for the same liquid. For example, 1 lb. of water at a temperature of 212° F. and at atmospheric pressure, requires the application of 966·6 British Thermal Units to convert it into 1 lb. of steam at the same temperature and pressure. Conversely, to condense 1 lb. of steam to 1 lb. of water, both being at 212° F. and 14·7 lbs. pressure per square inch, we must abstract from the steam 966·6 thermal units by contact with a cold body. This principle holds good for any liquid.

A refrigerating machine with steam as a working medium, would not be practicable unless the temperature of everything in connection with it was maintained above 212° F., but, there are many liquids which have, when compared with water, a very low boiling point; notably ether, sulphurous acid, ammonia, carbon dioxide, and ethylene. Each of these has been employed for purposes of refrigeration with more or less success; and all of them depend on the same principle—viz., the absorption or the giving out of their latent heat in converting the liquid to a gas, or *vice versâ.*

It is not necessary here, to enter into a discussion as to the relative merits of different liquids as refrigerating agents, but in practice, anhydrous ammonia is the agent generally used, and in a lesser degree, carbon dioxide. The necessary machinery is of itself, extremely simple; although, the details afford scope for a great amount of elaboration and ingenuity.

Carbon Dioxide as a Refrigerating Agent.—Carbon dioxide or carbonic anhydride, which is commercially known as "carbonic acid," is a colourless gas, and quite without odour when pure. It is under all circumstances perfectly innocuous, and has practically no effect on animal tissues or other bodies. It will, however, produce asphyxiation in animals when present in the atmosphere in quantities exceeding 25 per cent. by excluding oxygen from the blood. This gas may be very readily liquefied, either by diminishing its temperature or by increasing its pressure. This fluid has a specific gravity of about ·8, and can only remain in the liquid state when under considerable pressure, the pressure varying with the temperature of the liquid.* The moment the pressure is removed, the heat present in surrounding bodies, at once assists in the evaporation of the liquid carbon dioxide and the bodies themselves are consequently left in a colder condition than before the evaporation took place.

Elementary Refrigerating Apparatus.—Let us consider for a moment an elementary piece of apparatus in which refrigeration can take place. If we take two strong coils of piping and surround each with a vessel of water and then connect the two by means of a stop valve at their lower ends, as shown by the accompanying figure, we shall have a simple form of refrigerating machine. Suppose, that when the stop valve is closed, we charge the condenser coil with gas under liquefying pressure, by means of a force pump. The water surrounding the coil will absorb the heat which has been imparted to the gas by compression, and the condensed liquid will gradually accumulate at the bottom of the coil. On opening the stop valve, this liquid will run into the second or evaporating coil, and the pressure here being lower than is necessary for maintaining the liquid state of the material, evaporation will at once commence. The heat necessary for evaporating this liquid is absorbed from the water surrounding the evaporating coil, which will thereby become considerably reduced in temperature. To accomplish this result in practice three things are necessary :—(1) A compressor, to raise the pressure of the gas to whatever may be necessary for its liquefaction; (2) a surface condenser, to remove the heat generated by the mechanical work of compression; (3) an evaporating vessel where the liquid may re-evaporate into a gas, and absorb heat in the operation.

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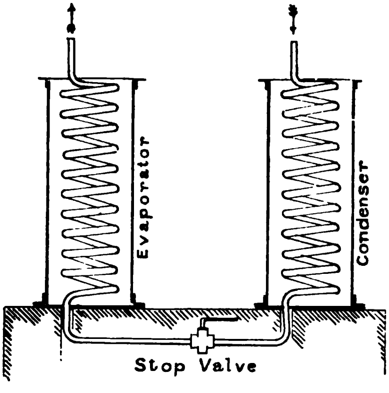
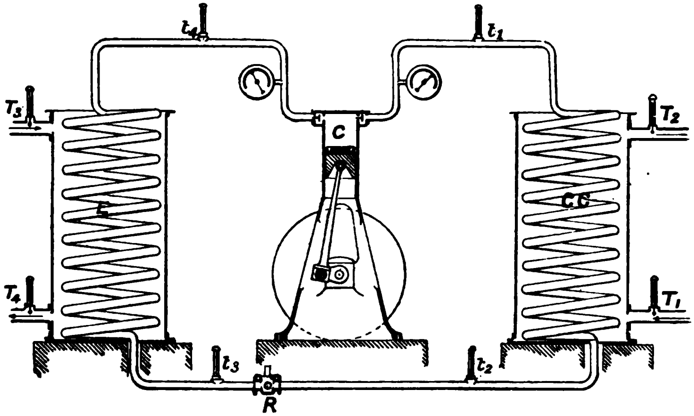
* Carbonic acid gas can only be liquefied by pressure when below 86° F. which is termed its critical temperature.



ELEMENTARY REFRIGERATING MACHINE.

Simple Refrigerating Machine.—The following figure of a simple refrigerating machine will explain the cycle of operations. C is a compressing pump delivering gas under pressure into the condensing coil C C, which consists of a strong worm of iron or copper piping immersed in a tank of water. R is a regulating-stop valve having a fine adjustment. E is the evaporator which consists of a coil of piping similar to the condensing coils. It is also immersed in a tank containing the water or other fluid to be cooled. The regulator R is closed, as soon as the pressure in the condenser has risen to that at which liquefaction can take place and the gas commences to condense on the inner surface of the coil C C. The drops of liquid descend and accumulate in the lower portion of this coil. The regulator is then opened, with the result, that a small quantity of liquid escapes into the evaporator. Now, since the compressor draws its supply of gas from the evaporator, the pressure in the evaporator must be less than in the condenser. Consequently, the liquid commences to boil, and absorbs heat for its transformation into a gas from the surrounding liquid. The temperature of this liquid is therefore naturally reduced by the operation. The liquid within the coil is entirely re-converted into a gas which ultimately finds its way to the compressor, and thus the cycle of operations is completed.



SECTIONAL DIAGRAM OF A SIMPLE REFRIGERATING MACHINE.

INDEX TO PARTS.

C	for Compressor.	E	for Evaporator.
C C	„ Condensing Coils.	t_1 to t_4	„ Thermometers.
R	„ Regulating Valve.	T_1 to T_4	„ Thermometers.

Suppose four thermometers be inserted into the pipes conveying the gas to and from the condenser and evaporator, as shown at t_1, t_2, t_3, t_4. It will be found that they do not register alike, for t_1 will show the highest temperature, then t_2 and t_3 will be some degrees lower, and t_4 will be lowest of all. Suppose now, that the liquids in the vessels surrounding C C and E be caused to circulate in the direction of the arrows, and that thermometers T_1, T_2, T_3, T_4, be placed on each of the inlet and outlet pipes. It will be found, that the temperature of the incoming water T_1 is lower than T_2 the temperature of the water going out of the condenser; also, that T_3 the temperature of the liquid entering the evaporator, is *higher* than T_4 its temperature as it leaves this vessel. This shows, that with respect to the gas or liquid within the coils of the condenser or evaporator, heat is lost in the condenser and gained in the evaporator. The amount of the former is represented by the difference between T_1 and T_2 multiplied by the weight of water passed through the condenser in pounds, and the latter may be expressed in terms of the difference between T_3 and T_4 multiplied by the weight of the fluid passing through the evaporator, and by the specific heat of this fluid.

If we could construct an ideal machine, in which the liquefaction of the gas was automatic, it would be found that the loss of heat in the condenser, measured in thermal units, was exactly equal to the gain of heat in the evaporator. The sensible heat gained and lost by the fluids surrounding the coils in the condenser and evaporator, respectively, would be the exact measure of the latent heat of the refrigerating medium, as abstracted in the condenser and returned in the evaporator. It is, however, necessary to change the physical condition of the gas between the evaporator and condenser, so that it can be liquefied in its passage through the latter vessel. Suppose that the pressures in both evaporator and condenser are the same and constant. In order to ensure condensation and liquefaction in the condenser, its temperature would have to be constantly maintained *below* that of the evaporator, a condition of things which is manifestly impracticable, since the evaporator is becoming colder with every repetition of the cycle of operations. This difficulty must therefore be met in another way. If we wish to liquefy any gas, it is necessary to bring its molecules closer together, and this can be accomplished either by *increasing* the pressure or by *decreasing* the temperature of the gas, or both. Now, since it is not in this case practicable to reduce the temperature, the only alternative is to raise the presusre by means of the pump already referred to, which draws the gas from the evaporator and delivers it at an increased pressure into the coils of the condenser. But in order to compress a gas, mechanical work must be performed upon it, and this work re-appears in the form of heat. The temperature of the gas after compression is therefore considerably higher than it was at the lower pressure on leaving the evaporator. This heat, in addition to the heat imparted in the evaporator, has to be abstracted and carried away by the cooling action of the water of the condenser.

As stated at the beginning of this Lecture, if we convert a unit weight of any liquid into a gas, we require the addition of a definite amount of heat, and to reconvert this gas into a liquid we require the abstraction of the same amount of heat, the amount being constant for any one liquid at a constant pressure and temperature. All gases do not require the same expenditure of energy to raise them to the same pressure, because they vary in what may be called their compressibility, and some gases occupy a smaller volume than others after an equal amount of compression. Carbon dioxide, for example, according to Regnault, only requires about 75 per cent. of the work necessary to produce the same amount of compression, as air or hydrogen. We can, by experiment, readily determine the exact pressure at which liquefaction will take place at any temperature; and knowing this, the machine can be designed of suitable strength to withstand the necessary pressure.

Owing to the difference in the power required to increase the pressure of different gases, it follows that the amount of heat imparted during compression must with some gases be greater than with others. This fact is of great importance in the selection of a suitable gas, and particularly so if cooling water be scarce. But whatever gas be employed, the pressure necessary to liquefy it must always be increased to a greater or less extent as the temperature of the cooling water rises.

Having considered the principles upon which an evaporative refrigerating machine depends for its action, we are now in a position to examine into the actual question of the interchangeability of heat and work. We can moreover at once establish a coefficient of efficiency for any refrigerating material.

Let, $L =$ Latent heat of evaporation of the refrigerating medium in B.T.U.

And, $H =$ Heat imparted during compression in B.T.U.

It follows, from what has been said, that the coefficient of efficiency will be $L \div H$, and, neglecting external losses, wL will represent the heat abstracted in the evaporator, whilst $H + wL$ will equal the heat added to the cooling water of the condenser, where w, is the weight in lbs. of the gas entering the condenser in a given time. It is, of course, impossible that a machine could work under such ideal conditions as we have assumed, since there must always be the effect of the high or low temperature of surrounding bodies to determine whether there will be a loss or gain of heat in one or other of the parts of the machine. For instance, it is almost certain that the evaporator will be colder than the atmosphere; in that case, no matter how carefully it may be insulated, there will be some conduction of heat and the net quantity of the heat abstracted from the liquid to be cooled can only be $(w\,\mathrm{L} - x)$, where x, is the amount of heat derived from outside sources. The amount of heat imparted to the cooling water in the condenser will therefore be $\mathrm{H} + w\,\mathrm{L} \pm y$; where y, is the heat lost or gained in the condenser due to the difference in temperature between it and its surroundings.

There is still another correction to be made to the above formula. When, as is usual, the evaporator is maintained at a very low temperature, a certain amount of heat must be imparted to it by the refrigerating liquid itself, as it is entering the evaporator in a comparatively warm condition. Thus, supposing there be t degrees difference in temperature between the condenser and evaporator, a unit weight of the refrigerating liquid will as it were import into the evaporator $t\,s$ thermal units; where s is the specific heat of the liquid in question. Therefore, if W be the weight of refrigerating liquid passing into the evaporator in a given time, the heat abstracted in the evaporator will be represented by the expression:—

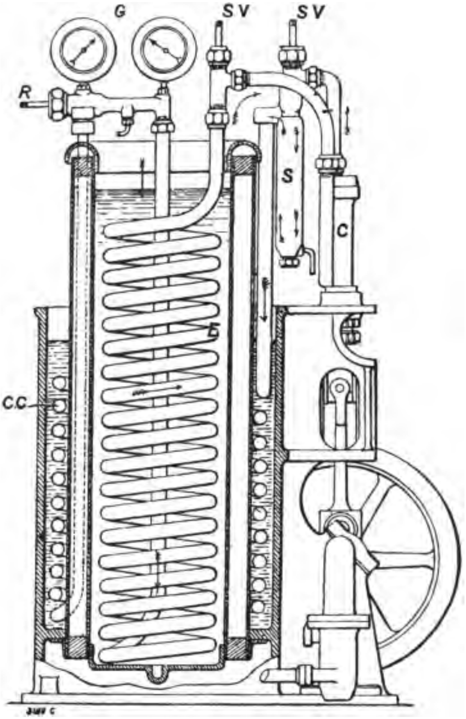
$$w\,\mathrm{L} - x - \mathrm{W}\,t\,s$$

Of course, $\mathrm{W}\,t\,s$ will not in practice amount to a great deal; but, as Professor Linde has pointed out, it must not be neglected in an exact calculation of the work performed by any refrigerating machine. If there be no leakage, then on the average W will be the same as w.

These formulæ cannot be applied with absolute certainty in practice, owing to the impossibility of making all the necessary corrections due to the gain or loss of heat in the various parts of the machine, and owing to the friction of the gas in constricted passages. But, with care, this gain and loss of heat can be very nearly accounted for in an ordinary machine, as manufactured for commercial purposes and working under the conditions of everyday practice.

Carbon Dioxide Refrigerating Plant.—One method of cooling buildings, &c., on a large scale, is to employ a strong brine obtained by dissolving sodium chloride or common salt in water. This brine is first cooled by passing it through the evaporator of a refrigerating machine, and then circulating it in pipes placed within the chambers which it is desired to cool.

In the accompanying figure all the essential parts are shown of a small refrigerating machine as manufactured by Messrs. J. & E. Hall, of Dartford, for cooling small provision stores, dairies, &c., where the pump may be conveniently driven by a belt. In larger machines, the evaporator is contained in a separate vessel and the compressor is driven by a compound- or triple-expansion engine; but, for the sake of compactness in this case, the evaporator is placed within its condenser, and the intervening space between them is carefully insulated by means of some non-conductor, such as hair felt or slagwool. The coils of



HALL'S CARBON DIOXIDE REFRIGERATING PLANT.

INDEX TO PARTS.

C for Compressor.
S ,, Separator.
CC ,, Condensing Coils.
R ,, Regulator.
G for Gauges.
E ,, Evaporator.
SV ,, Stop Valves.

piping which form the condenser and evaporator are welded into continuous lengths and so connected that all joints shall occur in accessible positions. These and all the other gas joints are made by inserting copper rings turned from the solid metal between a pair of flanges or union coupling. This form of joint has been found very satisfactory. The condenser casing, corliss frame and bearings for the compression pump, &c., are all made of cast iron in one casting. The compressor C is made of special hard bronze in order to ensure freedom from spongy places, while the suction and delivery valves are identical in shape and size so that they may be interchangeable. The compressors for larger machines are bored out of solid steel forgings. This ensures strength together with sound material. A true bore is also provided for the smooth working of the cupleathers with which the pistons are packed. The gland is made gas-tight by means of two U-leathers fitted over the compressor-rod and glycerine is forced between them under a somewhat greater pressure than that in the compressor. Any leakage which takes place is therefore of glycerine—outwards (which can be collected and used over again) and inwards—which both lubricates the interior of the compressor and fills up the clearance spaces, thereby increasing the efficiency of the machine. The superior pressure of glycerine in the gland is obtained by utilising the pressure in the condenser acting through a small intensifier, similar to those in use in hydraulic installations. Any glycerine which passes into the compresser, beyond what is necessary to fill up the clearance spaces is discharged with the gas through the delivery valves. In order to prevent this glycerine passing into the condenser coils, all the gas is delivered into a separator S and caused to impinge against the sides of this vessel. The glycerine adheres to its sides and drains to the bottom from which it may be drawn off from time to time, thus permitting the dry compressed gas to pass away by an opening at the top of the separator to the evaporator E.

One feature of these machines is the safety valve, which is fitted to the gas circuit immediately above the compressor, so that no harm can be done to the machine even if carelessly started with the stop valves closed. It consists of an ordinary spring safety valve, beneath which is a thin copper disc, designed to burst at a certain pressure. This disc can be made perfectly gas-tight, which could not be so easily accomplished by the spring safety valve alone. The latter only comes into play in the event of a rupture of the copper disc.

Anhydrous Ammonia as a Refrigerating Agent.—The most important advantages possessed by anhydrous ammonia as an agent for cooling purposes are, its freedom from the danger of explosion, its great latent heat and low pressure of vaporisation.* The latent heat of vaporisation of 1 lb. of carbonic acid at 0° F. is 123 units, and of ammonia 555 units, while the respective pressures in lbs. per square inch, at the same temperature, are 310 lbs. in the case of carbonic acid and only 30 lbs. with ammonia. It follows, therefore, that a carbonic acid plant must be constructed to deal with pressures of about 1,000 lbs. per square inch as against only 150 lbs. or so, in an ammonia machine. The exact pressures in each case are directly proportional to the temperature of the condensing water.

As already stated, this agent is more commonly used than any other, and in the United States of America, where refrigeration is applied to an extent unknown elsewhere, the machine generally employed is on the ammonia compression principle. It is similar to the carbonic acid machine in so far as the complete system consists of (1) a compression, (2) a condensing, and (3 an expansion part; moreover, the cycle of operations is exactly the same.

De La Vergne's Refrigerating Plant.—There are many different kinds of ammonia machines in use, but a general description of one of the best known and most extensively applied—viz., the "De La Vergne" as manufactured by Messrs. L. Sterne & Co., Ltd., of the Crown Iron Works, Glasgow, may be taken as a typical example.

In the following figure, A represents the ammonia compressor driven by a steam engine R. The gas which is returned from the expansion coil N, placed in the cooling chamber, enters the cylinder A by the pipe B, and after being compressed therein it is discharged, through the pipe C into a pressure tank D, together with a certain amount of sealing oil. Here, the oil, being heavier than the ammonia gas, naturally falls to the bottom, and the hot ammonia passes from the top of this tank by a pipe E to the condensers F; where, the cooling action of cold water trickling over the pipes causes the gas to liquefy. It then passes through pipes G, G to a header H, and from thence, to a storage tank I, which is simply a receptacle for holding a reserve supply of liquid ammonia. From this tank

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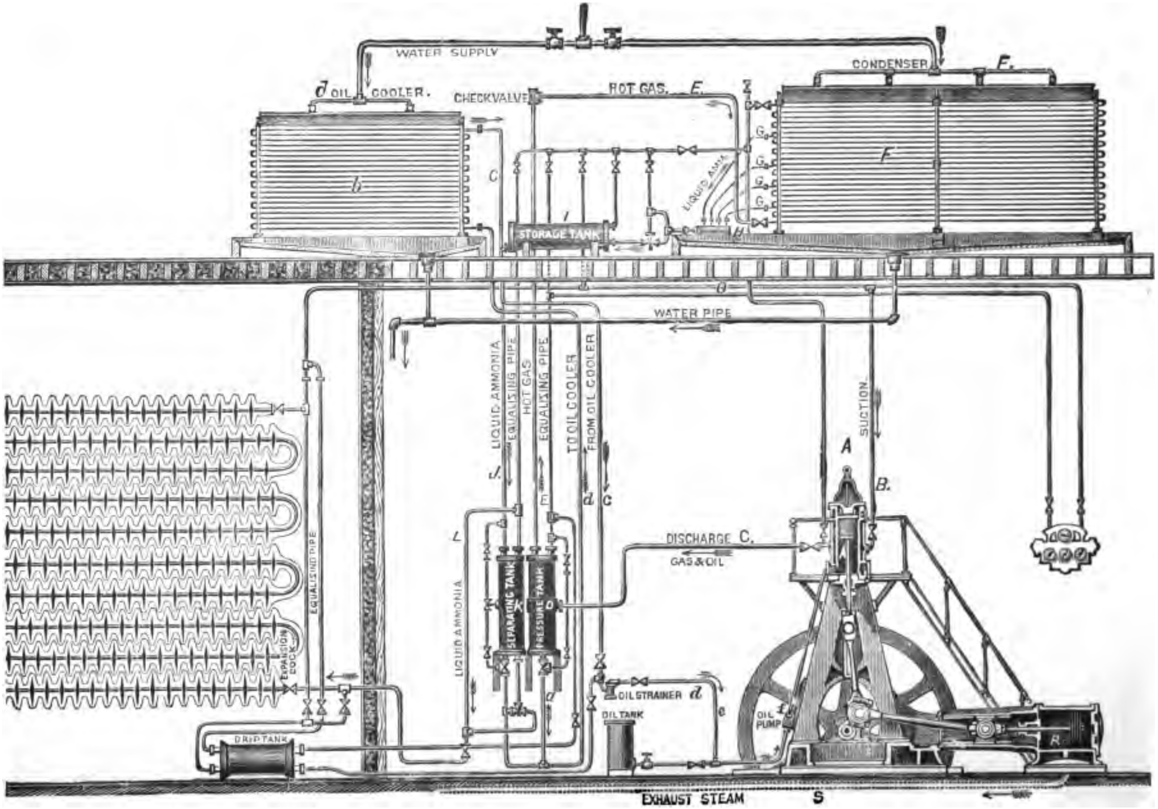
* A liquid with a high latent heat of evaporation need not necessarily be a good refrigerating agent, and *vice-versâ*. What is required is, that its specific heat should be low in proportion to the latent heat of evaporation. Or, we require as great a *difference* as possible between the latent heat of evaporation and the specific heat of the liquid multiplied by the range of temperature in the condenser and refrigerator.

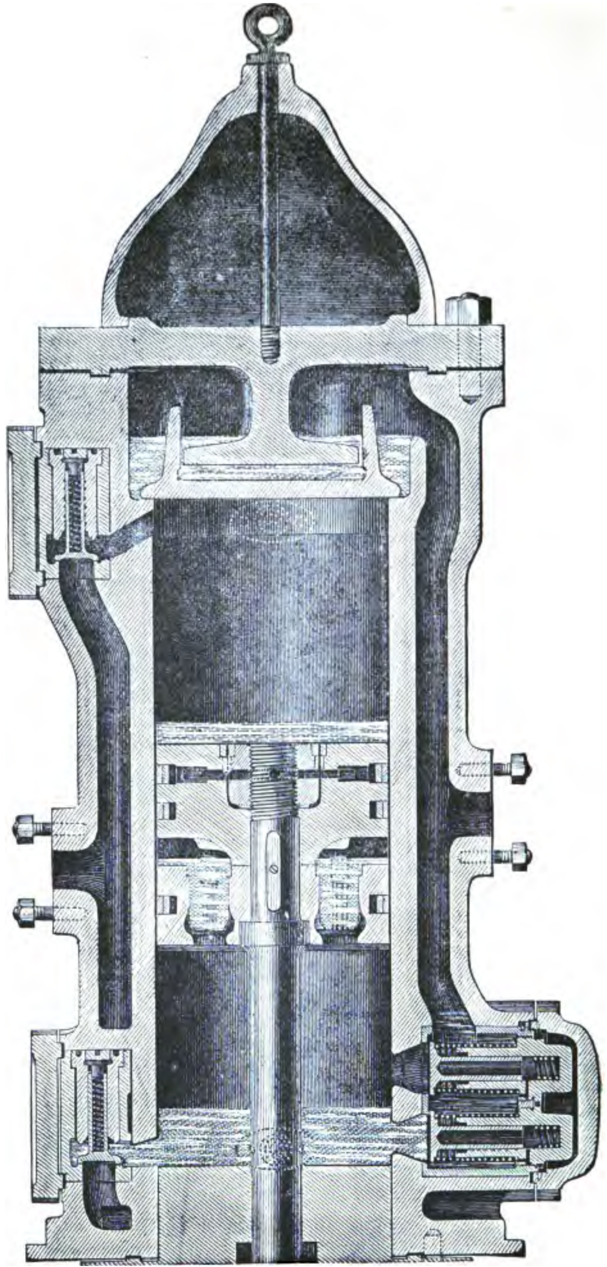


De La Vergne Refrigerating Plant. By L. Sterne & Co., Crown Iron Works, Glasgow.

it is conveyed by a pipe J to a separating vessel K, where any particles of oil that may have been carried over with the liquid are finally separated and the pure liquid ammonia is free to leave it by a pipe L to the expansion coils in the chambers to be cooled. The cock for admitting the ammonia to these coils can be regulated to any degree of minuteness. It thus serves to separate the high pressure from the low pressure part of the apparatus. Hence, the liquid ammonia on passing the expansion cock enters the cooling coils, which are maintained at a low pressure by the pumping action of the compressor. Here, it immediately flashes into gas and by abstracting from its surroundings the heat necessary to cause this change, the temperature of the room is lowered to any desired extent. After having thus done its work in the cooling chamber, the gas is returned to the compressor by a pipe B, to again undergo the same cycle of operations.

The sealing oil passes from the bottoms of the pressure and separating tanks D and K, by the pipes *a* and *d* to the oil cooler *b*; thence, by pipe *c* to the oil strainer *d* and the pipe *e* to the oil pump *f*; by which, it is again circulated through the compressor A.

De La Vergne's Double-Acting Compressor.—The accompanying figure is a section through a "De La Vergne" double-acting compressor, and shows the use of the oil seal. In all ammonia compressors, a certain amount of oil is required for lubricating purposes, and if the compressor be arranged in the ordinary way, the discharge valves at the lower end are placed either on the bottom or at the side, with the result that the oil is discharged *before* the gas. The oil ought, however, to be discharged *after* all the gas is gone; otherwise, re-expansion takes place which would entail a loss of efficiency. In the "De La Vergne" compressor this difficulty has been avoided in the following manner:—At the lower right-hand end of the compressor, two discharge valves are fitted into a side pocket, with the one fair above the other. On the down stroke, either of the valves or both may open until the piston covers the upper one, when only the lower valve is open to the condenser. In the further course of the piston and as soon as the lower valve is also closed, the upper one comes into direct communication with an annular chamber in the piston. This chamber has valves in its bottom side which open into it, as soon as all other inlets on the lower side of the piston are closed. The gas, therefore, first leaves the compressor and then the oil follows, thus permitting no gas to remain in the lower side after the completion of the down stroke. The effect of the oil seal is to make the compressor



De La Vergne's Double-Acting Compressor for his Refrigerator.

work with practically no clearance and thus a maximum of efficiency is obtained. The oil also serves to carry away a considerable amount of the heat of compression and to seal all the valves and stuffing-boxes.

Attention may now be drawn to a few of the details of the above plant. In the first place, it will be noticed that the ammonia condenser is not of the ordinary type where the coils containing the gas are usually submerged in a water tank, but they are of the open or atmospheric type. Here, water is kept constantly trickling over the condenser pipes, and the cooling action is therefore considerably assisted by the evaporation thereof from the surface of the pipes, which enables a maximum of condensation to be effected with a minimum of water supply. It also leaves all the pipes of the condenser open for examination and cleansing. This style of condenser is now coming into extensive use for the condensation of steam in large factories. In the second place, it will be seen that the refrigerating or cooling effect is caused by the direct expansion of the ammonia in pipes placed in the chamber to be cooled. This does away with the unavoidable loss of efficiency due to the use of a supplementary medium such as brine. It, however, necessitates very careful coupling up and jointing all the expansion coils, in order to prevent any leakage of the ammonia gas; more especially, in the case of a large plant where there may be as much as ten or more miles of piping in these cooling coils. In practice, however, these details have been so carefully worked out, that many hundreds of miles of such piping are constantly at work without giving the slightest trouble. Consequently, the old-fashioned method of brine circulation is not now so generally employed except on board ship, where there is a possibility of undue rocking or straining of the pipes and where it is considered advisable to use something that would cause no disagreeable odour in case of a broken pipe or joint.

In applying the refrigerating machine to the manufacture of ice, the simplest method is to place the expansion or cooling coils in a tank filled with brine or other non-congealable liquid, while the water to be frozen is placed in moulds of suitable size, which are then inserted into this brine until frozen. The purpose served by the brine in this case is to convey the heat from the water to the cooling pipes. It is therefore generally kept in slow circulation in order to ensure that the temperature shall be as uniform as possible throughout the tank. If ordinary well water be placed in the moulds, the resulting ice will contain so much air that it will be turned out of a milky white and opaque colour; but if the water whilst in the process of freezing be kept in slow motion by means of agitators, this air escapes, and a clear glassy ice is the result.

Another method of obtaining clear ice is to use distilled water. This has also the advantage of getting rid of any objectionable matter which might be in solution.

The Linde System of Refrigeration.—This system was first introduced into Germany in the year 1875 by Professor Linde, who was then one of the teaching staff at Munich University. In this country, however, prior to 1888, the principal cold-producing machinery, as manufactured for both land and marine purposes, was the simple cold air machine, in which refrigeration is produced by the compression, cooling when under compression by means of water and subsequent expansion of ordinary atmospheric air. These machines, although simple in construction and giving very good results, possess the disadvantage of requiring a large amount of power to work them in comparison with those employing more efficient refrigerating agents. Consequently, the former method has now very largely given way to one or other of the latter, of which the Linde system is one of the most successful, seeing that over 3,000 of these machines have been constructed up to September, 1897, representing an output, as rated by the capability of producing 69,200 tons of ice every twenty-four hours. In America the largest machine turned out upon any system is rated at 500 tons of ice per twenty-four hours.

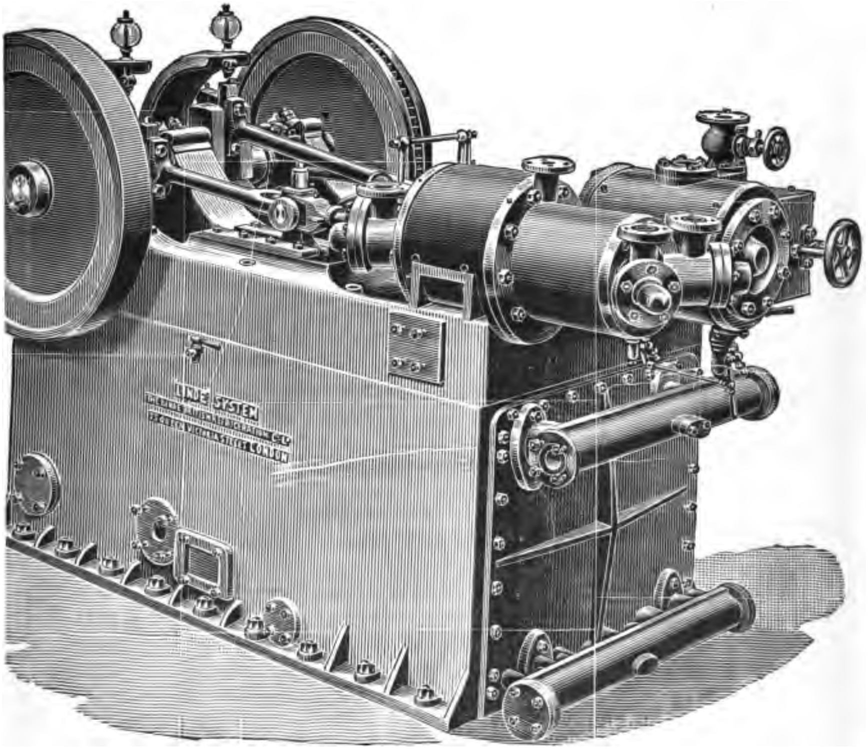
The Linde System of Refrigeration is identical in principle, and only differs in mechanical details from the De La Vergne previously described. It is, therefore, based on the evaporation of liquid anhydrous ammonia and the subsequent liquefaction thereof by means of mechanical compression, together with the cooling of the vapour thus formed, so as to enable it to be used over and over again. As will be seen from the accompanying illustration, the self-contained motive-power plant, as chiefly used on board ship, consists of a horizontal steam engine on the right, with a horizontal duplex compressor pump to the left, and an ammonia condenser in the sole plate. As far as the compressor is concerned, the chief differences between the Linde and the De La Vergne systems are :—

(1) That in the former a horizontal compressor is used instead of a vertical one in the latter case.

(2) That a special oil (not susceptible to change at any temperature attainable by the machine, which does not contain any acid or other deleterious matter, and which does not saponify when brought into contact with ammonia is used solely for lubricating purposes. Whereas, the oil used in the De La

Vergne system serves not only as a lubricant to the working parts, but also to partly carry away the heat of compression, and, further, to fill up the clearance spaces, as well as to seal the valves, glands, &c., so as to prevent the escape and consequent inconvenient smell of ammonia.

(3) In the Linde system a small quantity of liquid ammonia is introduced into the compressor during each suction stroke for the purpose of cooling the vapour of ammonia. This liquid ammonia evaporates during compression, and thus the heat due to compression, which would otherwise appear as sensible heat, is thereby absorbed and rendered latent in producing the change in the physical state of the liquid. The curve of compression is thus kept down as nearly as may be to the isothermal line, and


LINDE SYSTEM
THE LINDE BRITISH REFRIGERATION Cº LD
QUEEN VICTORIA STREET LONDON

SHIP REFRIGERATING MACHINERY ON THE LINDE SYSTEM.
BY THE LINDE BRITISH REFRIGERATING COMPANY.

the power required for compression is to this extent correspondingly reduced.

Apparatus for Transmitting the Cold Produced to the Chambers requiring Refrigeration.—The following general principles are adopted for transferring the cold generated by the refrigerating machinery to the chambers or rooms requiring to be cooled.

First Method.—An uncongealable solution of salt (chloride of sodium) in water is reduced by the refrigerating machine to a low temperature, and this liquor acts as transmitter of cold in one of the following methods:—

(*a*) The cold brine is constantly circulated from the brine refrigerator through pipes placed in the refrigerated chambers, and returned to the brine cooler. The result is that not only is heat abstracted from the air of the refrigerated rooms, but also a large degree of the moisture which may be present in them. This moisture is condensed on the exterior of the brine pipe systems either in the form of condensed water or hoar-frost. Suitable drip trays are provided, in order to prevent this moisture from falling upon the contents of the rooms. The circulation of air with this system is a moderate one, being produced merely by the differences between the temperatures prevailing near the brine-pipes and those in the lower parts of the rooms.

(*b*) The brine is cooled in a shallow rectangular open tank containing the evaporator coils. On the tank is mounted a number of slowly revolving transverse shafts, and on each shaft is fixed a number of parallel discs, partly immersed in the brine, the entire apparatus being placed in an insulated passage through which an air current is continually passed by a fan, in a direction parallel to the revolving discs. It will be seen, that as the discs revolve and are kept covered by a film of the refrigerated brine, the air passing between the disc-spaces becomes cooled, and produces a low temperature in any chamber or room into which it may be conducted through properly arranged air-trunks. As a rule the air is always taken back from the cold rooms, passed over the discs and returned to the cold rooms, and any required amount of fresh air is introduced by means of adjustable openings in the air-trunks, communicating with the outer atmosphere. In this instance, also, moisture may be removed from the refrigerated rooms and deposited in the brine contained in the trough. No accumulation of frost can take place, and the refrigerated surfaces are always perfectly active. The circumstance of all moisture being deposited in the brine necessitates either a periodical loss of the same or its re-concentration. The fan produces a very effective air circulation within the rooms to be cooled. This in most cases is extremely desirable, and, as will be readily understood, produces the most beneficial results.

Second Method.—Instead of using an uncongealable liquid as a bearer of cold, the refrigerator coils (in which the vaporisation of the liquid anhydrous ammonia takes place) are sometimes constructed with extra large surfaces, and placed either in the upper part of the rooms to be cooled, or in a separate chamber. In the latter case, a fan constantly circulates the air between this chamber and the refrigerated rooms. This is the system generally adopted on board ships, and has been found to be in all respects most satisfactory. In cases where the air temperature is not sufficiently high to cause a complete removal of the snow deposited on the ammonia-coils, the snow is thawed by the ammonia vapours themselves, the evaporator-coils being for the time used as a condenser. Occasionally the snow is thawed by a current of hot air taken from the outside.

Although all of these methods have been applied on an extensive scale, the system most strongly recommended in cases where its application is possible is the combination of revolving discs immersed in brine. There are no brine or ammonia pipes in the rooms; whilst the rapid air circulation by the fan is easily managed, and has been found in most cases to be requisite for obtaining a satisfactory result as to purity, dryness, and equable temperature in all the rooms.

Where circumstances require the refrigerated rooms to be at a distance from the refrigerating machine, it is generally most convenient to place bundles of brine pipes in each room; but even in such a case, in the event of a small amount of motive power being available close to such rooms, the system of revolving discs and fans can be readily applied, the brine being cooled in a refrigerator near the compressor, and conveyed to and from the disc tanks through insulated pipes.

A large beef-chilling plant on the Linde system was erected in the beginning of 1890 at the Woodside lairage of the Mersey Dock and Harbour Board. It is capable of chilling 660 carcases of beef, each weighing about 9 cwts., from 90 degrees Fahr. to 33 degrees in 17 hours. It consists of a horizontal compound tandem jet-condensing steam engine, which drives a double-acting Linde compressor at the rate of about 65 revolutions per minute, when supplied with steam at 120 lbs. pressure from a marine type boiler. The air-cooling apparatus consists of two disc tanks, placed above the chill rooms, at one end. Each disc system has its own fan, which draws the air from the top of each of the chill rooms, passes it over the discs, and drives it into the rooms at the opposite end to that from which it is withdrawn.

The ammonia condenser is placed in the compressor room, and is supplied with cooling water by a pump which takes its supply from a well, fed with the drainage water from the Mersey Tunnel. After passing through the ammonia condenser the water is used in the condenser of the steam engine. There are six chill rooms, each about 55 ft. long by 14 ft. wide, and about 13 ft. high. The walls of the rooms are built of brick, with air spaces. The floors are cement, and the ceilings are timber, covered with a layer of fine ashes. The air-cooling apparatus is contained in an insulated casing, which is so arranged as to cause the air to come in contact with the cooled surfaces of the discs.

The following is a list of some of the books and papers on Hydraulics and Hydraulic Machinery :—

Hydraulic Machinery, by R. Gordon Blaine. (E. & F. N. Spon, Ltd., London, 1897.)

Recent Hydraulic Experiments. Paper by Major A. Cunningham. Proc. Inst. C.E., vol. lxxi., p. 1.

Hydraulic Appliances at the Forth Bridge Works. Paper by E. W. Moir. Proc. Inst. C.E., vol. xci., p. 402.

The 160-Ton Hydraulic Crane at Malta Dockyard Extension Works. Paper by C. & C. H. Colson. Proc. Inst. C.E., vol. cxiv., p. 289.

On Machine Tools and other Labour-Saving Appliances Worked by Hydraulic Pressure. Paper by R. H. Tweddell. Proc. Inst. C.E., vol. xxiii., p. 64.

The Barry Dock Works, including the Hydraulic Machinery and the Mode of Tipping Coal. Paper by J. Robinson. Proc. Inst., C.E., vol. ci., p. 129.

The Distribution of Hydraulic Power in London. Paper by E. B. Ellington. Proc. Inst. C.E., vol. xciv., p. 1.

Hydraulic-Power Supply in London. Paper by E. B. Ellington. Proc. Inst. C.E., vol. cxv., p. 220.

Forging by Hydraulic Pressure. Paper by R. H. Tweddell. Proc. Inst. C.E., vol. cxvii., p. 1.

Hydraulic-Power Supply in Towns. (Glasgow, Manchester, Buenos Ayres, &c.) By Edward B. Ellington, M. Inst. M.E. (Proc. Inst. Mechanical Engineers, Glasgow Meeting, 1895).

Water-Pressure Engines for Mining Purposes. Paper by H. Davie. Proc. Inst. M.E., 1880, p. 245.

Hydraulic Flanging of Steel Plates. Paper by Easton & Anderson. Proc. Inst. M.E., 1882, p. 518.

Portable Hydraulic Drilling Machine. Paper by M. Berrier-Fontaine. Proc. Inst. M.E., 1887, p. 72.

Lifts in the Eiffel Tower. Paper by A. Ansaloni. Proc. Inst. M.E., 1889, p. 350.

Hydraulic Packing Presses. Paper by C. Hopkinson. Proc. Inst. C.E., vol. xcix., p. 275.

The Treatment of Steel by Hydraulic Pressure and the Plant Employed for the purpose. Paper by W. H. Greenwood. Proc. Inst. C.E., vol. xcviii., p. 83.

Hydraulic Propulsion. Paper by S. W. Barnaby. Proc. Inst. C.E., vol. xxvii., p. 83.

A Hydraulic Pumping Engine. Paper by H. D. Pearsall. Proc. Inst. C.E., vol. cvi., p. 292.

Hydraulic Work in the Irawadi Delta. Paper by R. Gordon. Proc. Inst. C.E., vol. cxiii., p. 276.

Centrifugal Pumps, Turbines, and Water Motors, by C. H. Innes, M.A. (The Technical Publishing Co., Ltd., Manchester.)

Lifting and Hauling Appliances in Portsmouth Dockyard. Paper by J. T. Corner. Proc. Inst. M.E., 1892, p. 295.

Hydraulic Power and Hydraulic Machinery, by Henry Robinson. (Chas. Griffin & Co., London, 1896.)

Hydraulic Motors, Turbines, and Pressure Engines, by G. R. Bodoner. (Whittaker & Co., London, 1889.)

A Treatise on Hydraulics, by M. Merriman. (John Wiley & Sons, New York, 1895.)

Hydrostatics and Elementary Hydrokinetics, by G. M. Minchin. (Oxford Press, Oxford, 1892.)

Hydraulic Machinery for Glasgow Harbour Tunnel, as Constructed and Erected by the Otis Elevator Co., New Yonkers, New York. See *Engineering,* May and June, 1895.

Paper on *Refrigerating Apparatus,* by Prof. Carl Linde, of Munich. *Journal of the Society of Arts,* 9th March, 1894.

Paper on *Experiments on a Two-Stage Air Compressor,* by John Goodman, Wh.Sc. Proc. Inst. C.E., vol. cxxviii., and also *The Practical Engineer,* 30th July, 1897.

Compend of Mechanical Refrigeration, by J. E. Siebel. (H. S. Rich & Co., Chicago, 1895.) In this book there are several important tables and other valuable information regarding refrigerators.

Adaptation of Hydraulic Power in the Manufacture of Iron and Steel. Paper by James L. Biggart. *Journal of the West of Scotland Iron and Steel Institute,* 1893.

Application of a System of Combined Steam and Hydraulic Machinery to the Loading, Discharging, and Steering of Steamships. Paper by A. Betts Brown, F.R.S.E. Proc. Inst. Nav. Arch., 1890.

LECTURE XXXIV.—QUESTIONS.

1. Explain the fundamental principles upon which a refrigerating machine works. Note specially what becomes of the different quantities of heat generated and absorbed.
2. Sketch the essential parts of a refrigerator, and describe its action.
3. What are the advantages which a vapour possesses over a permanent gas, such as air, for refrigerative purposes?
4. What are the requirements of an economical medium for use in a refrigerator?
5. Sketch and explain the plant required for producing cold by means of carbon dioxide.
6. Explain the reasons that have led to the adoption of anhydrous ammonia in most modern refrigerators, and mention some of the properties of this vapour.
7. Sketch and describe any well-known arrangement for refrigerating, using anhydrous ammonia.
8. Explain and illustrate some of the ways of communicating cold to a chamber from a refrigerator.

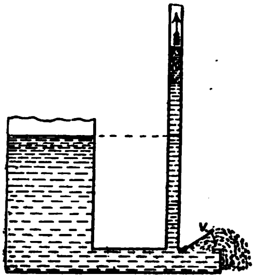
APPENDIX TO PART VI.

ON

HYDRAULICS AND HYDRAULIC MACHINERY.

The following description of the Hydraulic Ram and examples and questions which we propose to incorporate in an extension of Part VI. in a future edition are now printed here in the hope that they may be of immediate use to students.

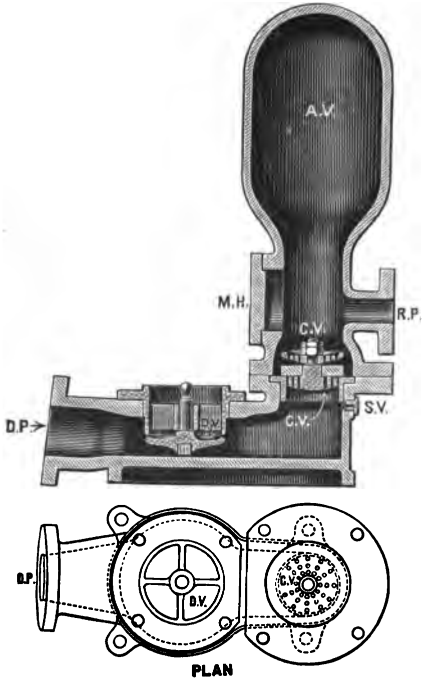
Hydraulic Ram.—This apparatus was invented about 100 years ago by a Frenchman named Montgolfier. It is one of the simplest, most durable, and efficient machines for raising water to a greater height than the source of supply. The energy stored up in water descending from a comparatively low elevation is utilised to raise part of the same water to a much higher level of from three to thirty times the vertical height of the original fall. The principle upon which the apparatus works will be understood from a consideration of the accompanying figure. If the valve V be held down firmly on its seat, and the left-hand vessel be filled with water to a certain height, it will rise to the same level in the right-hand open pipe. If the valve be now released for a short time, water will flow under the action of gravity along the horizontal passage and escape at the open valve with a velocity proportional to the square root of the "head" or vertical height of the free surface above the valve. On suddenly closing the valve, the kinetic energy of the moving water will be partly spent in raising the right-hand column to a greater height than the free surface of the water in the left-hand vessel. Now, if we introduce a check valve at the foot of the long column, so as to prevent this water from falling down again, and an air vessel to act as a cushion, we can repeat the



PRINCIPLE OF THE HYDRAULIC RAM.

operation continually, so as to produce a flow of water up the pipe.

The machine, as made and supplied by the Glenfield Company of Kilmarnock, is illustrated by a vertical section and plan in the following figure. Water flows from a cistern, tank, pond, or dam, through a cast- or wrought-iron pipe, technically called the drive pipe D P, to a hollow casting containing two valves. The



HYDRAULIC RAM, BY THE GLENFIELD COMPANY, KILMARNOCK.

first of these is named the escape or dash valve D V, which opens downwards, and the other the check valve C V, which opens upwards. Over the latter is fixed an air vessel A V, having a manhole door M H to the left and a delivery pipe, which is technically termed a rising pipe R P, to the right. If the apparatus and all the pipes are duly connected to the supply and delivery tanks, and the dash valve D V be held up, until the water from the source of supply has filled not only the drive pipe D P, but also risen through the check valve C V and rising pipe R P to nearly the same level as the free surface in the supply tank, the whole will remain motionless or in a static condition. If we now depress the dash valve D V, and then let it go, the machine will immediately begin to work, and continue to work automatically without any attention or even oiling for years, until stopped by some accident or by the wearing out of one or both of the valves. Of course, the supply of water must be maintained, so that the drive pipe is always kept full. This pipe should not be throttled in any part, and the weight of the dash valve must be so carefully adjusted, that it will just overcome the static pressure—*i.e.*, drop from its seat—and permit water to escape thereat. Then, the acceleration produced by gravity on the water coming down the drive pipe very soon produces a greater pressure on the dash valve than that due to the mere static pressure. This increased force suddenly raises it again to its seat, when the kinetic energy which has been imparted to the water lifts the check valve and forces some water into the air vessel. Whenever this kinetic energy has been spent, the compressed air in the air vessel, together with the weight of the check valve, causes it to close, and immediately thereafter the dash valve automatically opens. The same cycle of operations takes place over and over again, the air in the air vessel gets more and more compressed, and water rises higher and higher in the rising or delivery pipe, until it issues as a continuous stream from its mouth into the cistern or receiving tank. From this tank it may be drawn off at pleasure for all the various uses of a mansion-house or farm steading, &c.

The air vessel plays two important parts in each cycle of the operations of this interesting and useful apparatus. (1) The air contained therein acts as a cushion by minimising the water hammer action, which would otherwise stress the various parts, and tend to break the joints. (2) The air acts as a store of energy by taking up, during its compression, a part of the kinetic energy of the water, and then giving out the same gently, thus producing a constant flow of water through the delivery pipe. If the vertical height of the column of water in the rising pipe be about 34 feet above the check valve, the pressure per square inch on the upper surface of this valve will be one atmosphere, or, say, 15 lbs. on the square inch, and the air in the air vessel will be compressed to nearly that pressure, and therefore occupy about half its original volume. If the column be 68 feet high in the delivery pipe, the pressure on the valve will be about 30 lbs. on the square inch, and the air in the air vessel will occupy one-third of its original volume, and so on. Hence, it is necessary to proportion the size of the air vessel to the vertical height through which the water has to be forced in the delivery pipe. Besides this, air becomes absorbed by water, and in a short time the air vessel, if small, would become entirely filled with water. The air vessel may, however, be kept charged with air in a very simple manner by the introduction of a snifter valve S V, screwed into the ram casing, immediately below the check valve. In its simplest, and probably its most efficient form, it consists of a brass plug with a very small hole drilled through its axis. Every time that water is forced through the check valve a very small quantity also passes through this tiny opening in the snifter valve; and each time that the check valve is forced down upon its seat a rebound or reaction of the water takes place, and produces a partial vacuum immediately underneath the check valve. Consequently, a little air is forced into this vacuum by the atmospheric pressure, and this air is carried up into the air vessel at the next stroke or pulsation, thus keeping up the necessary supply for effecting a continuous flow of water into the receiving tank. If everything about this machine is thoroughly tight and in good working order, and the valves are made of the best proportions and weights, an efficiency of from 80 to 90 per cent. can be obtained therefrom, and it has been found possible to work it with a minimum driving head of only three feet. The several causes for loss of efficiency are:—

(1) The friction of the water passing along the drive pipe D P, and the casing of the apparatus.

(2) The weight and friction of the dash valve D V, which has to be lifted at each stroke or pulsation.

(3) The weight and friction of the check valve C V, which has also to be lifted at each stroke.

(4) The slip of the check valve C V—*i.e.*, a slight quantity of water may slip back past this valve when in the act of closing.

(5) The friction of the water passing along the rising pipe R P.

(6) Any defects of tightness in the faces of the dash and check valves.

EXAMPLE I.—If 1,000 lbs. of water pass per minute through the drive pipe under a head of 6 feet, and 60 lbs. of water are delivered into the receiving tank, which is 87 feet above the check valve, what is the efficiency?

$$\text{Efficiency} = \frac{\text{work got out}}{\text{work put in}} \text{ (in the same time).}$$

$$\therefore \qquad \text{Efficiency} = \frac{87 \times 60}{6 \times 1000} = \cdot 87$$

Or, **Efficiency = 87 per cent.**

If the length of the drive and rising pipes be considerable, and if there be many bends and much throttling of the passages, then the efficiency will thereby be reduced to a considerable extent. By a simple modification of the ram shown in the illustration, river or impure water may be made to raise spring or pure water; the two waters are separated by a diaphragm, and the pumping action actuates two valves, the one being a suction and the other a delivery valve.

EXAMPLES ON HYDRAULICS.

Example I.—A cylindrical tank, 6 feet in diameter and 10 feet deep, is filled with water; find the bursting pressure round the base of the tank, and the pressure on its base.

Answer.—The bursting pressure round the base is measured by the intensity of the fluid pressure on any small area of the curved surface infinitely near to the base. This pressure will be exactly equal to that on the base. Hence, the question resolves itself into finding the intensity of the pressure on the base.

$$\therefore \left.\begin{array}{r}\textbf{Bursting pressure}\\ \textbf{round the base}\\ \textbf{of tank}\end{array}\right\} = \textit{Pressure per square inch on base.}$$

$$\text{,,} \quad \text{,,} \quad = a\,\mathrm{H\,W}.$$

$$\text{,,} \quad \text{,,} \quad = \frac{1}{144} \times 10 \times 62{\cdot}5 = \mathbf{4{\cdot}34\ lbs.\ per\ sq.\ in.}$$

$$\text{Again,} \quad \left.\begin{array}{r}\textbf{Total pres-}\\ \textbf{sure on base}\end{array}\right\} = \left\{\begin{array}{l}\textit{Area of base in sq. ins.} \times \textit{pressure}\\ \quad\textit{per sq. in.}\end{array}\right.$$

$$\text{Or,} \quad \text{,,} \quad \text{,,} \quad = \frac{22}{7} \times 36 \times 36 \times 4{\cdot}34 \text{ lbs.} = \mathbf{17{,}678\ lbs.}$$

Example II.—A circular water tank is 20 feet in diameter and 25 feet deep. It is constructed of 6 rings of cast-iron plates. Find the *total* stress on any vertical section of the bottom row of plates made by a plane passing through the axis of the cylinder, neglecting any assistance afforded by the flanges or connection with the bottom plate. (S. and A. Exam., 1888.)

Answer.—It has been proved in Lecture XXIX. that when a cylindrical shell is subjected to internal fluid pressure, the *total* stress in the material along any section made by a plane containing the axis of the cylinder is equal to the total fluid pressure on either side of that part of the plane intercepted within the cylinder.

Hence, total stress in material of bottom row of plates = total fluid pressure on vertical plane through the axis of the cylinder at the bottom row of plates.

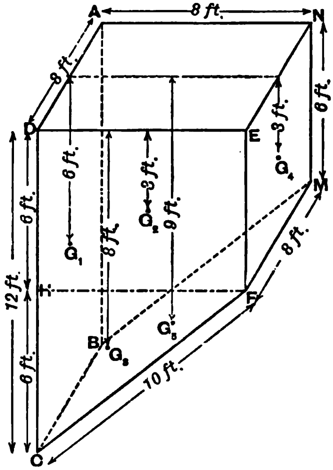
Since the breadth of each ring $= \frac{25}{6} = 4\frac{1}{6}$ ft.; therefore, depth of *c. g.* of bottom ring $= H = 25 - \frac{1}{2} \times 4\frac{1}{6} = 22{\cdot}9\dot{6}$ ft.

$$\therefore \left.\begin{array}{l}\textbf{Total stress in material}\\ \textbf{along section at bottom}\\ \textbf{row of plates}\end{array}\right\} = A\,H\,W,$$

$$\text{,,} \quad \text{,,} \quad = (20 \times 4\tfrac{1}{6}) \times 22{\cdot}9\dot{6} \times 62{\cdot}5 \text{ lbs.}$$

$$\text{,,} \quad \text{,,} \quad = \mathbf{119{,}357 \text{ lbs.}}$$

EXAMPLE III.—How is the pressure of water on a given area immersed in it ascertained? A water tank, 8 feet long and



PRESSURE ON SIDES OF TANK.

8 feet wide, with an inclined base, is 12 feet deep at the front and 6 feet deep at the back, and is filled with water. Find the pressure in lbs. on each of the four sides, and on the base; water weighing $62\frac{1}{2}$ lbs. per cubic foot.

ANSWER.—The total fluid pressure on any area immersed in the fluid is given by the formula—$P = A\,H\,W$.

Where A = Area of surface exposed to the fluid pressure,
H = Depth of centre of gravity of immersed area below free surface of fluid,
W = Weight of a cubic unit of fluid.

The shape and dimensions of the tank will be readily seen from the figure on previous page.

(*a*) *To find the total pressure on the front* A B C D.

Here, $A = AD \times DC = 8 \times 12 = 96$ sq. ft.

$H = \frac{1}{2}$ depth $DC = 6$ ft. $W = 62\frac{1}{2}$ lbs. per cubic ft.

$\therefore$ **Pressure on front A B C D** $= A\,H\,W$,

„ „ $= 96 \times 6 \times 62\frac{1}{2} =$ **36,000 lbs.**

(*b*) *To find the total pressure on the back* E F M N.

Here, $A = FM \times MN = 8 \times 6 = 48$ sq. ft.; $H = \frac{1}{2}EF = 3$ ft.

$\therefore$ **Pressure on back E F M N** $= A\,H\,W$,

„ „ $= 48 \times 3 \times 62\frac{1}{2} =$ **9,000 lbs.**

(*c*) *To find the total pressure on base* C B M F.

Before we can find the area of the base, we must know its length C F. From F draw F H parallel to E D, and therefore perpendicular to D C. Then C H F is a right-angled triangle whose sides are $FH = DE = 8$ ft., and $HC = DC - DH = DC - EF = 6$ ft.

$$\therefore \quad CF = \sqrt{HM^2 + ME^2} = \sqrt{8^2 + 6^2} = 10 \text{ ft.}$$

$$\therefore \quad A = CF \times CB = 10 \times 8 = 80 \text{ sq. ft.}$$

Again, the depth of the *c. g.* of the base C B M F is clearly—

$$H = \tfrac{1}{2}(DC + EF) = 9 \text{ ft.}$$

$\therefore$ **Pressure on base C B M F** $= A\,H\,W$,

„ „ $= 80 \times 9 \times 62\frac{1}{2} =$ **45,000 lbs.**

(*d*) *To find the total pressure on either side* C D E F or A B M N.

In this case it is perhaps best to divide the trapezoidal area C D E F into two figures whose centres of gravity can be easily determined. Thus, the line F H divides the side C D E F into a rectangle D E F H, and a triangle F H C. Then the total pressure on C D E F is equal to the sum of the pressure on D E F H and F H C.

Area of $DEFH = 8 \times 6 = 48$ sq. ft.

And, $\left.\begin{array}{r}\textit{Depth of c. g. of}\\ \textit{area}\ DEFH\end{array}\right\} = \frac{1}{2}EF = 3$ ft.

∴ *Pressure on* D E F H = A H W

„ „ $= 48 \times 3 \times 62\frac{1}{2}$ lbs.

Again, *Area of* F H C $= \frac{1}{2}$ H F × H C $= \frac{1}{2} \times 8 \times 6 = 24$ sq. ft.

The *c. g.* of triangle F H C is at a distance of $\frac{1}{3}$ of H C below the horizontal F H, and therefore at a distance of $6 + \frac{1}{3}$ of $6 = 8$ feet below D E.

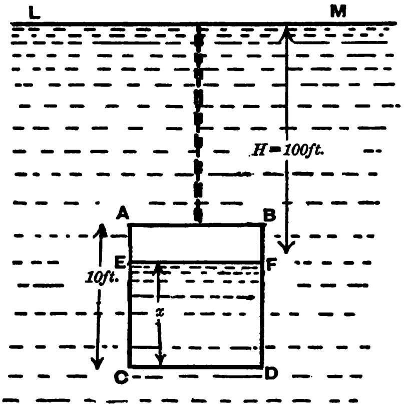
∴ *Pressure on* F H C = A H W,

„ „ $= 24 \times 8 \times 62\frac{1}{2}$ lbs.

∴ **Pressure on side C D E F or A B M N** } $= 48 \times 3 \times 62\frac{1}{2} + 24 \times 8 \times 62\frac{1}{2}$ lbs.,

„ „ $= 48 \times 62\frac{1}{2} \times (3 + 4) =$ **21,000 lbs.**

Example IV.—A cylindrical vessel, 10 feet long, open at one end and closed at the other, forms a diving bell. It is lowered into water with its open end downwards until the surface of the water in the cylinder is at a depth of 100 feet. Find how far the water has risen in the cylinder, and the pressure of the contained air. (Take the height of the water barometer as 34 feet.)



Answer.—

Let H = Depth of surface of water in bell = 100 feet.

„ h = Height of water barometer = 34 feet.

„ l = Length of cylinder forming bell = 10 feet.

„ x = Height that water rises in bell.

Before the bell is immersed in the water the pressure of the contained air is simply that due to the atmosphere. After immersion the pressure will be greater than that of the atmosphere by an amount due to a head of water of H feet.

Assuming, then, that the air in the bell has been compressed according to Boyle's Law ($p\,v =$ a const.), we get :—

$$\left.\begin{matrix}\textit{Press. of compressed air}\\ \times\ \textit{Vol. of compressed air}\end{matrix}\right\} = \left\{\begin{matrix}\textit{Press. of atmosphere}\\ \times\ \textit{Vol. of bell.}\end{matrix}\right.$$

$$\therefore \qquad \frac{\textit{Vol. of compressed air}}{\textit{Vol. of bell}} = \frac{\textit{Press. of atmosphere}}{\textit{Press. of compressed air}}.$$

Since the bell is of uniform cross sectional area throughout, we get:—

$$\frac{\textit{Vol. of compressed air}}{\textit{Vol. of bell}} = \frac{l-x}{l}.$$

$$\therefore \qquad \frac{l-x}{l} = \frac{h}{\mathrm{H}+h}.$$

$$\therefore \qquad x = \frac{\mathrm{H}\,l}{\mathrm{H}+h} = \frac{100 \times 10}{100+34} = 7{\cdot}46 \text{ feet.}$$

The pressure of the air in the bell when immersed is equal to the pressure due to a depth of $(\mathrm{H}+h)$ feet of water.

$$\therefore \quad \textbf{Pressure of air in bell} = a\,(\mathrm{H}+h)\,\mathrm{W} = \frac{1}{144} \times 134 \times 62{\cdot}5$$

$$,, \qquad ,, \qquad = 58{\cdot}16 \textbf{ lbs. per sq. inch.}$$

Example V.—Given two simple bucket pumps, each having a stroke of 1 foot, and cross area of bucket, 20 square inches. Suppose everything perfectly air tight, and the supply pipe 20 square inches area in one case, and 10 square inches area in the other. Neglecting friction, you are to compare the tensions on the pump rods at ends of first up stroke in each case, supposing the bucket to be 24 feet above free surface of the water in the well when at the bottom of its stroke. The supply pipe reaches to the under surface of bucket when the latter is at the bottom of its stroke.

Answer.—

Let P = Pressure in lbs. per *sq. ft.* on *under* surface of bucket at end of first up stroke.

,, A = Area of bucket $= \frac{20}{144}$ sq. ft.

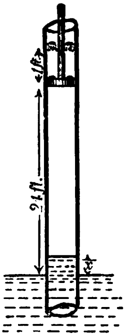
,, h = Height of water barometer = 34 ft.

,, W = Weight of 1 cub. ft. of water = $62\frac{1}{2}$ lbs.

,, x = Height water rises in suction pipe for first up stroke.

First Case.—*Lifting or suction pipe having an area equal to that of the bucket.*

Then at end of first up stroke, we get:—



First Case.

$$\left.\begin{array}{l}\textit{Pressure of air on}\\ \textit{upper surface of}\\ \textit{bucket,}\quad .\quad .\end{array}\right\} = \textit{Atmospheric pressure}$$

$$\text{,,}\qquad\text{,,}\qquad = \mathrm{A}\,h\,\mathrm{W}\ \textit{lbs.}\quad .\quad .\quad .\quad (1)$$

$$\left.\begin{array}{l}\textit{Pressure on under}\\ \textit{surface of bucket,}\end{array}\right\} = \mathrm{P\,A} = \left\{\begin{array}{l}\textit{Atmos. pressure}\\ \textit{- pressure of}\\ \textit{small column, } x,\\ \textit{of water}\end{array}\right.$$

$$\text{,,}\qquad\text{,,}\qquad = \mathrm{A}\,(h - x)\,\mathrm{W}.$$

$$\text{And,}\qquad \mathrm{P} = (h - x)\,\mathrm{W}.\quad .\quad .\quad .\quad (2)$$

Since there is exactly the same quantity of air between the bucket and the surface of the water in the pipe at the end of the stroke as there was before the stroke commenced, we may apply Boyle's Law to determine the *volume* of air under the bucket at end of stroke.

$$\therefore\qquad \mathrm{P\,A} \times (25 - x) = \mathrm{A}\,h\,\mathrm{W} \times 24.$$

Substituting, $\mathrm{P} = (h - x)\,\mathrm{W}$, from equation (2), we get:—

$$(h - x)\,(25 - x) = 24\,h.$$

Since, $h = 34$ ft., we get, by substitution, and multiplication:—

$$x^2 - 59\,x + 34 = 0.$$

$$\therefore\qquad x = \frac{59 \pm 57{\cdot}83}{2}\ \text{ft.}$$

The minus sign in the numerator of the fraction on the right-hand side of this equation is the only one admissible.

$$\therefore\qquad x = \frac{1{\cdot}17}{2} = {\cdot}58\ \text{ft. } \textit{or} = 7\ \text{inches, nearly.}$$

Hence,

$$\text{Tension in pump rod} = \left\{\begin{array}{l}\textit{Press. on upper surface of bucket}\\ \quad\textit{- Press. on under surface.}\end{array}\right.$$

$$\text{,,}\qquad\text{,,}\qquad = \mathrm{A}\,h\,\mathrm{W} - \mathrm{A}\,(h - x)\mathrm{W} = \mathrm{A}\,x\,\mathrm{W}.$$

$$\text{,,}\qquad\text{,,}\qquad = \frac{20}{144} \times {\cdot}58 \times 62{\cdot}5 = 5{\cdot}04\ \text{lbs.}$$

SECOND CASE.—*Lifting or suction pipe having an area equal to half that of the bucket.*

The symbols denoting the same quantities as before, we get:—

Pressure of air on upper surface of bucket . . } $= A\,h\,W$ lbs. . (3)

Pressure on under surface of bucket . . . } $= P\,A$

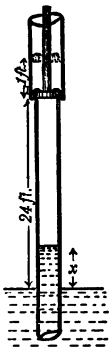
„ „ $= A\,(h - x)\,W$. (4)

Vol. of air between bucket and surface of water at beginning of stroke . } $= \frac{1}{2} A \times 24$

„ „ $= 12\,A$ *cub. ft.*

Vol. of air between bucket and surface of water at end of up stroke . . } $= \frac{1}{2} A \times (24 - x) + A \times 1$

„ „ $= \frac{1}{2}\,(26 - x)\,A$ *cub. ft.*



SECOND CASE.

∴ By Boyle's Law, we get:—

$$(h - x)\,W \times \tfrac{1}{2}\,(26 - x)\,A = h\,W \times 12\,A.$$

Substituting $h = 34$, and simplifying, we get:—

$$x^2 - 60\,x + 68 = 0$$

$$\therefore \qquad x = 1{\cdot}15 \text{ ft. } or = 13{\cdot}8 \text{ inches, nearly.}$$

Hence,

Tension in pump rod = A × W.

„ „ $= \frac{20}{144} \times 1{\cdot}15 \times 62{\cdot}5 = 10$ **lbs.**

EXAMPLE VI.—Show, with the aid of sectional sketches, the construction of the ordinary hydraulic lifting jack. If, in such a machine, the mechanical advantage of the lever or handle is 12 to 1, and the diameter of the lifting ram is 2 inches, while the diameter of the plunger is $\frac{7}{8}$ of an inch, what weight can be lifted theoretically when a pressure of 50 lbs. is applied to the lever handle? (S. & A. Exam., 1891.)

ANSWER.—The hydraulic jack has been fully described and illustrated in the author's *Elementary Manual*, Lecture XX.

Let D = Diameter of ram = 2 inches.
„ d = „ plunger = $\frac{7}{8}$ inch.
„ a = Mechanical advantage of lever = 12 : 1.
„ P = Effort applied at end of lever = 50 lbs.
„ W = Weight raised.

If Q denotes the pressure on the plunger, caused by the effort P applied at the end of the lever, then :—

$$Q = P\,a.$$

But,

$$\frac{W}{Q} = \frac{\text{area of ram}}{\text{area of plunger}} = \frac{D^2}{d^2}.$$

$$\therefore \quad \frac{W}{P\,a} = \frac{D^2}{d^2}.$$

This is a formula giving the relation between W, P, a, D, and d, which is equally true for the hydraulic press.

Substituting the above values in this formula, we get :—

$$\frac{W}{50 \times 12} = \frac{2^2}{(\frac{7}{8})^2}.$$

$$\therefore \quad W = 3{,}134 \text{ lbs., } \textit{or} = 1{\cdot}4 \text{ tons, nearly.}$$

QUESTIONS ON HYDRAULICS AND HYDRAULIC MACHINERY.

Fluid Pressure.

1. State Archimedes' principle. A cylindrical can is 6 inches in diameter and 30 inches deep, and is required, when empty, to stand in a bath of water 30 inches deep without being lifted up. To what weight must the can be loaded, the weight of a cubic foot of water being 62½ lbs.? *Ans.* 30·679 lbs.

2. A rectangular tank, 4 feet square, is filled with water to a height of 3 feet. A rectangular block of wood, weighing 125 lbs., and having a sectional area of 4 square feet, is placed in the tank, and floats with its sides vertical and with this section horizontal. How much does the water rise in the tank, and what is now the pressure on one vertical side of the tank? *Ans.* 2 inches; 1,253·47 lbs. (S. and A. Exam., 1892.)

3. One side of a reservoir has a slope of 12 vertical to 5 horizontal; what is the whole amount of the pressure of the water against 50 feet of its length, when the depth of the water is 12 feet? *Ans.* 243,750 lbs.

4. In an empty dock the water is level with the sill at the lowermost edge of the dock gate, the level of the water on the opposite side of the gate being 10 feet above the sill. The dock gate is 10 feet wide, find the pressure in pounds on the dock gate. If the water were at a level of 5 feet in the dock, the level outside being the same as before, how much would the pressure on the gate be relieved? *Ans.* 31,250 lbs.; 7,812·5 lbs.

5. Prove that when a thin spherical shell is exposed to the bursting pressure of gas or liquid the stress in the material is half as great as that within the curved surface of a thin cylindrical shell exposed to the like pressure, each shell being of the same thickness and diameter. (S. and A. Exam., 1891.)

Hydraulic Pumps.

1. Describe a force pump for supplying water to the accumulator of hydraulic cranes. Sketch a section through the plunger and valves.

2. Sketch and describe a force pump having a solid plunger, showing the construction of the valves. The diameter of the plunger is 2½ inches, and it is driven by a crank 2 inches in length making 30 revolutions per minute, find the number of cubic inches of water pumped up in 5 minutes. *Ans.* 2945·25 cub. ins.

3. Sketch a combined plunger and bucket pump with index of parts, explaining its use and action, also sketch its application to the lifting of water from deep mines. Suppose a pump raises 5,000 gallons every half minute from a depth of 600 feet with 30 per cent. loss in system, what is the horse-power required? *Ans.* 1874·4 H.P.

4. Describe and illustrate by a longitudinal section, and such other views as may be necessary, the construction and action of a double-acting pump and its valves, supposing the pump cylinder to be of 3½ inches internal diameter, and to work at a pressure of 700 lbs. per square inch. Of what materials would the several parts be constructed? (S. and A. Exam., 1893.)

5. A vertical single-acting pump has to elevate water 50 fathoms. The bore of the pump is 6 inches; stroke, 6 feet; number of up strokes, 10 per minute. Find (*a*) the pressure per square inch on the pump backet when it is at the bottom of its stroke; (*b*) the weight of water discharged per minute; (*c*) the horse power of the engine required to drive the pump, supposing 30 per cent. of the engine-power to be lost by friction, &c. Sketch an arrangement of the kind. *Ans.* (*a*) 130·28 lbs. per square inch; (*b*) 736·31 lbs.; (*c*) 9·56 H.P.

6. Make a sketch of a centifrugal pump. What form should you give to the blades? Explain the theory of the action of such a pump.

7. Describe and show, with the aid of necessary sketches, the construction of the "Pulsometer." Describe how it works, and indicate the contrivances introduced to promote the steady flow of water and to prevent sudden shocks upon the apparatus. Is the pulsometer an economical arrangement for raising water? Give reasons for your answer. What, if any, are its advantages over the ordinary piston pump? (S. and A. Hons. Exam., 1896.)

8. Give any method with which you are acquainted whereby the valves in a pumping engine may be relieved from the shock due to the inertia of the water in the mains.

9. A pump for exhausting air from a receiver has a solid piston and one valve in the casing. Describe, with a sketch, the construction of the pump, and explain the nature of the improved form known as Sprengel's pump, where a small portion of mercury forms the piston, and no valve is required. (S. and A. Exam., 1889.)

HYDRAULIC PRESS.

1. Sketch in section the cylinder, ram, and leather collar of an hydraulic press. Explain the principle of the press and the manner in which the escape of water is prevented. Example—The sectional area of the plunger of the force pump is $\frac{1}{50}$ that of the ram, and the leverage gained by the pump handle is 12 to 1, find the pressure on the ram when a force of 60 lbs. is exerted at the end of the pump handle. *Ans.* 36,000 lbs.

2. Describe, with a sectional sketch, a hydraulic press where the ram is actuated in both directions. Show the position and forms of the cup leathers.

3. The return stroke in an hydraulic press is often accomplished by forming the ram like a piston with a very large piston-rod. Sketch in longitudinal section such a press, showing the arrangement of the leathers. What will be the relative speeds of the forward and return strokes of the ram when the larger and smaller diameters are 15 inches and 14 inches respectively, the pumps for the supply of water running at the same speed in both cases? (S. and A. Adv. Exam., 1894.)

4. In some hydraulic presses a single valve, held down by a lever and weight, is used both to indicate and relieve the pressure. Sketch the valve in position and explain its action.

HYDRAULIC CRANES AND HOISTS.

1. Make a section through the cylinder of an hydraulic crane with two powers as applied for raising weights. Give a general description of the crane, and explain the mode of working it by referring to your sketch.

2. Describe the general construction of the lifting apparatus in the hydraulic crane, making a sketch of the cylinder and ram. Show the

method of obtaining *two* powers from a single cylinder. In what way is the *pulley principle* applied in such cranes?

3. In an hydraulic crane with two powers show (1) the apparatus for lifting the weight, (2) the method of slewing or rotating the jib of the crane.

4. Make a sketch of a 10-ton hydraulic jib crane in which the lifting cylinder is carried in the pillar or post of the crane. What would be the diameter of ram required in the arrangement you adopt, supposing water to be supplied to the crane at a pressure of 700 lbs. per square inch, neglecting friction, &c.? (S. and A. Exam., 1891.)

5. What is the object of a relief valve in an hydraulic crane, and where is it placed? Sketch a longitudinal section through the cylinder and ram of a crane working at two powers. Explain the mode of action. (S. and A. Exam., 1888.)

6. Describe, with sketches, some form of hydraulic lift, and the manner in which it is worked. (S. and A. Exam., 1889.)

HYDRAULIC APPLIANCES, &C.

1. Sketch a section through an hydraulic lifting jack and hydraulic bear, and describe the manner in which the pressure exerted on the handle is transmitted to the ram.

2. A 4-ton hydraulic lifting jack has a lifting ram of 2 inches in diameter, and a pump plunger of 1 inch in diameter. The jack is worked by a lever handle, the leverage being 16 to 1. What pressure must be applied at the end of the handle in order to lift a load of 25 cwts., if the efficiency of the machine is 80 per cent.? Make a vertical section of the jack, showing the valves and the mode of connecting the lever with the pump plunger. How can the weight be lowered slowly and regularly without jerks? (S. and A. Adv. Exam., 1894.)

3. Explain, with the aid of a sectional sketch, the action and construction of the hydraulic jack. How is the pressure taken off and the load slowly lowered? If the ram is 2 inches in diameter, the pump plunger $\frac{7}{8}$ inch, and the mechanical advantage of the handle 10, what is the total mechanical advantage, neglecting friction. (S. and A. Adv. Exam., 1897.) See Author's *Elementary Manual.*

4. What are the advantages in forging large masses of steel by hydraulic pressure over the same operation performed by the steam hammer? Show clearly, with the assistance of the necessary sketches, the method employed in hydraulic forging presses for bringing the ram or pressing surface rapidly back from the work after each application of the pressure. (S. and A. Hons. Exam., 1896.)

5. Describe clearly and show with sketches the construction and action of any one form of portable hydraulic riveting machine with which you are acquainted. Show clearly the valves and connections by which the pressure is applied to close the rivet, and how the pressure is released and the tool withdrawn from the rivet head when the riveting is completed. How is the water pressure conveyed to the machine? (S. and A. Hons. Exam., 1895.)

6. What is the use of an intensifier or intensifying accumulator in the working of hydraulic machinery? Sketch such an apparatus, and explain fully its principle and construction: give also one example of the application of the intensifier to hydraulic machinery. (S. and A. Adv. Exam., 1896.

7. Describe a vertical breast water-wheel of large size. How should the buckets be made, and in what manner is the driving pressure of the wheel communicated to the shafting which it actuates? State the reasons for the construction to which you refer.

8. Describe, with a sketch and index of parts, a vertical flow turbine.

9. Sketch the wheel and case of an inward flow turbine for a fall of 50 feet; 8 cubic feet of water per second. Calculate the diameters and breadths of the wheel, the number of revolutions per minute, and the size of the shaft. (S. and A. Hons. Exam., 1897.)

10. Water is flowing fairly uniformly in a nearly rectangular channel through a meadow; it has fallen from a height of 30 feet. State how you would proceed to find, with a very rough approximation to accuracy, the water power of the fall. What measurements would you find it necessary to make? (S. and A. Adv. Exam., 1897.)

11. Sketch a water ram, and explain its action.

12. Describe the siphon, and explain its action. In a siphon on a large scale, employed for drainage, a stand pipe, communicating with an air pump, is attached to the top of the siphon tube, also the stand pipe is fitted with one or more taps for the admission of air. For what purposes are the air pump and the air taps respectively required? (S. and A. Exam., 1887.)

13. Explain the advantage to be derived from making the length of stroke of an hydraulic engine adjustable. Describe and give sketches of a construction for this purpose.

14. An escape valve, loaded partly by a weight and partly by a spring, is fitted to a main conveying water under pressure, and is required to open automatically when the water pressure rises above a certain amount. Sketch and describe the construction of such a valve when arranged on the double beat principle, and explain clearly the hydrostatic principle involved therein. (S. and A. Exam., 1890.)

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